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Alexander Ziwet

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# TREATISE

ON THE

# CALCULUS OF VARIATIONS.

ARRANGED WITH THE PURPOSE OF INTRODUCING, AS WELL AS ILLUSTRATING, ITS PRINCIPLES TO THE READER BY MEANS OF PROBLEMS, AND DESIGNED TO PRESENT IN ALL IMPORTANT PARTICULARS A COMPLETE VIEW OF THE PRESENT STATE OF THE SCIENCE.

BY

LEWIS BUFFETT CARLL, A.M.

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# PREFACE.

THIRTY years have now elapsed since the appearance of the treatise on the Calculus of Variations by Prof. Jellett, which, although it had been preceded by the smaller work of Woodhouse in 1810, and of Abbatt in 1837, is justly deemed the only complete treatise which has ever appeared in English. But all the works named have long since been out of print, and are now so rare as not to be found in the majority of the college libraries of the United States. Moreover, even Prof. Jellett's treatise can no longer be regarded as complete, since its author had not read the memoirs of Sarrus and Cauchy relative to multiple integrals, while the contributions of Hesse, Moigno and Lindelöf, and Todhunter were subsequent to the publication of his work. It should be added, also, that all the memoirs and contributions just named are contained in works which are likewise out of print, and are now almost as difficult of access to the general reader as is that of Prof. Jellett.

These considerations first led the author to undertake the preparation of the present treatise, in which he has endeavored to present, in as simple a manner as he could command, everything of importance which is at present known concerning this abstruse department of analysis.

In the execution of this design the following method has, so far as possible, been pursued: When a new principle is to be introduced for the first time, a simple problem involving it is first proposed, and the principle is established when re-

quired in the discussion of this problem. This having been followed by other problems of the same class, the general theory of the subject is finally given and illustrated by one or two of the most difficult problems obtainable; after which another principle is introduced in like manner.

Although the view taken of a variation is that of Profs. Airy and Todhunter, and the methods of varying functions are those of Jellett and Strauch, still all the other leading conceptions and methods have, it is hoped, been explained with sufficient fulness to enable the reader to follow them when they occur in other works.

The history of the subject is also briefly given in the last chapter, it being believed that the proper time for the presentation of the history of any science is after the reader has become familiar with its principles, as it can then, by the use of some technical terms, be accomplished more fully within a given space.

To aid the non-classical reader, the use of Greek letters has, with the exception of two, whose use is now universal, and which are explained, been avoided, except in references, or in such passages as may be omitted without serious loss. Attention is also called to the words *brachistochrone* and *parallelepipedon*, which are in this work spelled according to their derivation. The correct orthography of the former had been previously adopted by Moigno and Todhunter, and it is hoped that it may be sufficient to call the attention of Greek scholars to the latter.

One of the great obstacles to the preparation of the present treatise has been the difficulty of procuring the authorities which it was necessary to consult; and the author would here return his thanks to the officers of his Alma Mater, Columbia College; to Dr. Noah Porter, the President, and Mr. A. Van Name, the Librarian, of Yale College; and to Mr. Walter M. Ferris, of Bay Ridge, L. I., for the extended loan of rare works which could not be found in other libraries, or

if found, could not be had at home for that careful study which they in many cases demanded. The author is also greatly indebted to Lieut. Fred. V. Abbot, U.S.A.; to M. S. Wilson, Ph.B., to Prof. P. Winter, and to the late A. Sander, Ph.D., all of the Flushing Institute, for valuable assistance in the examination of French and German works. But the greater part of the assistance which the author has received was rendered by his youngest brother, who, in addition to aiding in the examination of many works, recopied the manuscript for the printer, and subsequently undertook, in conjunction with the author himself, the proof-reading of the entire publication.

It having been found necessary to publish the present treatise by subscription, the author, supported by President Barnard of Columbia College, Prof. J. H. Van Amringe of the same, Joseph W. Harper, Jr., and others, issued an appeal to the public, which shortly elicited the following subscriptions, the copies being placed at four dollars each:

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Warned by the experience of others, the author was con-

vinced from the first that he could hope to derive no pecuniary profit from a work like the present. But if it is now possible that there may accrue to him some small financial return, this possibility is due to the liberality of his publishers, who, although consulted late, and knowing the unremunerative character of the work offered, proposed voluntarily to undertake its publication upon terms more favorable than those which he had been endeavoring to secure.

The acknowledgments of the author are due also to his printer, S. W. Green's Son, for the general excellency of the proof furnished, and especially for his uniform readiness to do, without regard to trouble, whatever was indicated as tending to render the work more correct in minor points.

But while the author has, in the particulars mentioned, received much assistance from friends, to whom he would return his unfeigned thanks, he deems it but just to himself to say that he has never enjoyed the acquaintance of any one who had made the Calculus of Variations the subject of extensive study, and has consequently been obliged to depend solely upon his own judgment and the various works which he has consulted.

It is not therefore believed that the present treatise can be entirely free from mathematical errors; and hence the author would respectfully ask his readers, and especially those among them who may have given previous attention to this subject, to indicate any points in which his methods or results appear erroneous, or any places in which misprints may have been allowed to pass unnoticed.

L. B. CARLL.

FLUSHING, QUEENS CO., N. Y., July 8, 1881.

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# CALCULUS OF VARIATIONS.

## CHAPTER I

### MAXIMA AND MINIMA OF SINGLE INTEGRALS INVOLVING ONE DEPENDENT VARIABLE.

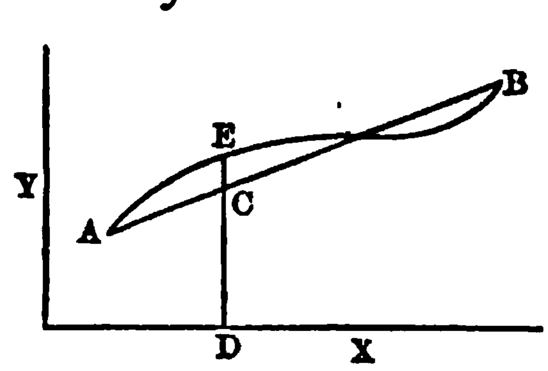
### SECTION I.

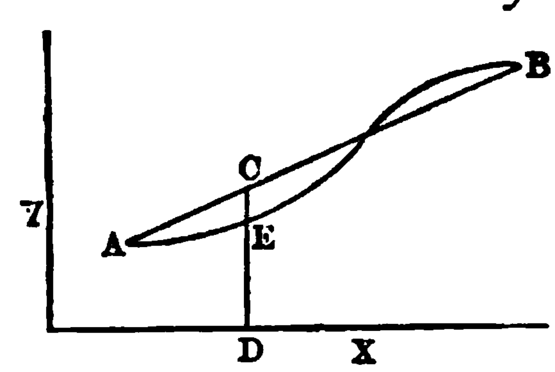
*CASE IN WHICH THE LIMITING VALUES OF X, Y, Y', ETC., ARE GIVEN.*

### Problem I.

1. *Suppose it were required to find the shortest plane curve or line which can be drawn between two fixed points.*

Let $ACB$ be the required line, which is of course straight, and $AEB$ any other line derived from the first by giving




indefinitely small increments to any or all of its ordinates, while the corresponding values of $x$ remain unaltered. Then the line $ACB$ must be shorter than the line $AEB$.

This remark would be equally true if the changes in the

ordinates of $AB$ had not been made indefinitely small; but then, even if the second line were shown to be longer than the first, we could not be certain that some third line, lying a little nearer the first, might not be shorter than either. Thus it will be seen that questions may arise which require an investigation of that increment which a curve would receive, not from any change in the values of $x$, nor in the values of the co-ordinates of the fixed extremities, but from indefinitely small changes in the values of $y$ throughout the whole or a portion of the curve; thus altering in a slight degree the functional relation which previously subsisted between $x$ and $y$.

**2.** Now the general expression for the length of any plane curve between two fixed points is

$$l=\int_{x_0}^{x_1}\sqrt{dx^2+dy^2}, \tag{1}$$

in which the suffix 1 relates to the upper, and 0 to the lower limit of integration, and this expression cannot be integrated so long as $y$ is an unknown function of $x$.

Hence, in determining the increment which will result to a curve from an indefinitely small change in its form, we shall be concerned with two species of small quantities: first, those changes which $x$ and $y$ undergo as we pass from one point to another indefinitely near or adjacent on the same curve, which are denoted by $dx$ and $dy$, these being necessary for the general expression of $l$ in (1); and secondly, that change which $y$ undergoes as we pass from a point on one curve to a point on another curve indefinitely near or adjacent, the value of $x$ being unaltered. These latter quantities are called variations, and are denoted by the Greek letter $\delta$, delta, or $d$. Thus $\delta y$ is read, the variation of $y$; $\frac{\delta dy}{dx}$, the variation of $\frac{dy}{dx}$, etc.

As another illustration of the difference between these two

classes of quantities, we might say that $dy$ as used in (1) is the difference between two consecutive states of the same function of $x$, while $\delta y$ is the difference between two consecutive or adjacent functions taken for the same value of $x$. The use of this symbol $\delta$ is due to Lagrange, and while it prevents confusion, it also suggests the character of the variation as a species of differential. It is plain that we can vary the form of a curve which terminates in two fixed points in any manner we please, by simply giving suitable changes to its ordinates without varying its abscissæ, and we shall therefore at present ascribe no variation to the independent variable $x$, but simply to the dependent $y$ or to its differential coefficients with respect to $x$.

**3.** Resuming equation (1), we will now show how to find $\delta l$, or that increment which $l$ would receive, not from any change in the limits of integration, but from an inappreciably small alteration in the value of $y$ as a function of $x$. We shall in general put $y'$ for $\frac{dy}{dx}$, $y''$ for $\frac{d^2y}{dx^2}$, etc. Then we have

$$dx^2 + dy^2 = \frac{dx^2 + dy^2}{dx^2}\,dx^2 = (1 + y'^2)\,dx^2;$$

hence (1) becomes

$$l = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\,dx. \tag{2}$$

It will be seen that $y$ does not occur directly or explicitly in the last equation; but since $y'$ represents the natural tangent of the angle which a tangent to the curve at any point makes with the axis of $x$, it is clear that the form of this curve can be also altered at pleasure by giving suitable variations to the slopes of these tangents, and that if these variations be

indefinitely small, the remarks that have been made regarding $\delta y$ will be equally true regarding $\delta y'$.

Equation (2) may be written

$$l = \int_{x_0}^{x_1} V dx,$$

where

$$V = \sqrt{1 + y'^2}.$$

Now in $V$ change $y'$ into $y' + \delta y'$.

Then the new state of $V$, being denoted by $V'$, may be developed by the extension of Taylor's Theorem, thus:

$$V' = V + \frac{dV}{dy'}\delta y' + \frac{1}{2}\frac{d^2V}{dy'^2}\delta y'^2 + \frac{1}{1\cdot 2\cdot 3}\frac{d^3V}{dy'^3}\delta y'^3 + \text{etc.},$$

where, following the analogy of differentials, we write $\delta y'^2$, $\delta y'^3$, etc., for $(\delta y')^2$, $(\delta y')^3$, etc. Hence, if we call $V' - V$, $\delta V$, we have

$$\delta V = \frac{dV}{dy'}\delta y' + \frac{1}{2}\frac{d^2V}{dy'^2}\delta y'^2 + \frac{1}{6}\frac{d^3V}{dy'^3}\delta y'^3 + \text{etc.},$$

in which $\frac{dV}{dy'}$, etc., are the partial differential coefficients of $V$ with respect to $y'$.

But

$$l = \int_{x_0}^{x_1} V dx;$$

whence, if we change $V$ into $V'$, $dx$ remaining unaltered, and denote the new state of $l$ by $l'$, we shall have

$$l' = \int_{x_0}^{x_1} V' dx,$$

and calling $l' - l$, $\delta l$, we arrive at the equation

$$\delta l = \int_{x_0}^{x_1} \left\{ \frac{dV}{dy'} \delta y' + \frac{1}{2} \frac{d^2V}{dy'^2} \delta y'^2 + \frac{1}{6} \frac{d^3V}{dy'^3} \delta y'^3 + \text{etc.} \right\} dx. \quad (3)$$

**4.** Before proceeding it may be well to advert to the theory of maxima and minima, as developed by the differential calculus.

A function is said to be a maximum when its value is greater, and a minimum when its value is less, than that which it would have if any or all of its variables should receive indefinitely small increments, either positive or negative. Thus while the greatest value of a function, if not infinite, is always a maximum, it does not follow that every maximum is the greatest value of which the function is capable. Neither is the greatest value in every case the only maximum. The foregoing remarks apply equally to a minimum, it being only necessary in either case to compare the supposed maximum or minimum state of the function with the value of the states which immediately precede and succeed it.

Taking, for simplicity, a function of a single variable, this state is determined and comparison effected as follows: Let $f$ be any function of $x$ and constants, and change $x$ into $x + h$. Then if we develop $f'$, the new state of the function, by Taylor's Theorem, and subtract the original state, we shall have

$$f' - f = \frac{df}{dx} h + \frac{1}{2} \frac{d^2f}{dx^2} h^2 + \text{etc.}, \quad (4)$$

$h$ being either positive or negative.

We shall denote this series by $S$. Then, if $f$ is to be a maximum or minimum, $f' - f$ must be negative in the former case and positive in the latter, independently of the sign of $h$. But if no differential coefficient in $S$ become infinite, and we make $h$ indefinitely small, the sign of $S$ will either depend upon that of its first term, which cannot be independent of $h$,

or, if that term reduce to zero, upon the sign of the first that does not.

Now if this term be of an odd order, its sign would be affected by any change in that of $h$; but if of an even order it would not, since $h$ must be real. Hence any value of $x$ which would render $f$ a maximum or minimum must at least satisfy the equation $\frac{df}{dx} = 0$, and the roots of this equation furnish us with trial values of $x$, which, when substituted in the remaining terms of $S$, must render the second term negative for a maximum and positive for a minimum, or must fulfil the same condition for some other term of an even order, having reduced those which preceded it to zero; and we must reject those values of $x$ which do not satisfy these conditions.

It may also be useful to observe that $\frac{df}{dx}$ does not represent the exact ratio of the increments of $f$ and $x$, $dx$ being infinitesimal, but merely the limit of that ratio; that is, the value toward which it may be made to approach to within any assignable limit, but which it can never actually equal, it being meaningless to say that $dx$ ever really becomes zero. Or, better, we may regard $\frac{df}{dx}$ as merely a function derived from $f$ by certain algebraic methods which accord with the rules of differentiation; and the same remarks will apply to the higher differential coefficients of $f$.

Hence, since these coefficients are entirely independent of any increment which $f$ actually receives, we may, without altering any of them, replace $h$ in (4) by $dx$, $\delta x$, or any other infinitesimal we please.

**5.** If the roots of the equation $\frac{df}{dx} = 0$ comprised all the values of $x$ which could render $f$ a maximum or minimum, still, since $f$ might be capable of several maxima or minima,

we would have to determine which maximum would be the greatest, or which minimum the least; although the determination would in general be easy enough. But the equation in question does not give all the required values of $x$. For, if any of the differential coefficients in (4) become infinite, the reasoning of the last article will no longer hold true. In fact, it is well known that $f$ can become a maximum or minimum when its first differential coefficient is infinite, or when the same is finite while the second is infinite. These instances are examples of what are often termed failing cases of Taylor's Theorem—although, strictly speaking, the theorem does not fail at all, only the development becomes useless from its indeterminate character, and that not from any imperfection in the theorem itself, but owing to the existence of such conditions as to render impossible an entirely finite development of the form required.

**6.** Since the value of $h$ in (4) is altogether independent of its coefficients, and might be replaced by $dx$, $\delta x$, or any other symbol we please, it is clear that the form in which we have expressed $\delta l$ in (3) is analogous to that of $S$ or $f'-f$, except that each term in $\delta l$ is multiplied by $dx$, and is under an integral sign, and that the function taken is one of $y'$ and constants, among which $x$ is reckoned.

Considering the first term of that expression, viz.,

$$\int_{x_0}^{x_1} \frac{dV}{dy'}\,\delta y' dx,$$

we see that by taking $\delta y'$ indefinitely small throughout the curve we may ultimately render this term greater than the sum of the others, unless, indeed, that integral becomes zero for all possible values of $\delta y'$; it being understood that the variation of any quantity is to be always infinitesimal as compared with that quantity. It is also clear that if we change the sign of $\delta y'$ throughout the integral—that is, of each $\delta y'$,

leaving its minute numerical value unaltered—we shall also change the sign of the above integral, while the sign of the succeeding integral in (3) will remain unchanged.

**7.** From an examination of the figures, Art. 1, it will be seen that if $ACB$ be the minimum line between two fixed points, and we draw a second in any manner we please by giving infinitesimal variations to $y'$, we may also draw a third line by giving to $y'$ variations numerically equal but of opposite sign. Then, since $ACB$ is a minimum, $l'-l$ or $\delta l$ must be positive; $l'$ being the length of either of the lines $AEB$.

Hence, from the reasoning of the last article, we must have

$$\int_{x_0}^{x_1} \frac{dV}{dy'}\,\delta y' dx = 0,$$

since otherwise $\delta l$ could not be of invariable sign, as its sign would be the same as that of the above integral, which could be made to vary by changing that of $\delta y'$. Moreover, the second term in $\delta l$, viz.,

$$\frac{1}{2}\int_{x_0}^{x_1} \frac{d^2V}{dy'^2}\,\delta y'^2 dx,$$

must become positive; or if it reduce to zero, some other term of an even order must become positive for all values of $\delta y'$, all the preceding terms having reduced to zero.

But, as in the differential calculus, the foregoing is based upon the supposition that none of the differential coefficients of $V$ in (3) become infinite within the limits of integration, or, in other words, that $V'-V$ is throughout these limits capable of a finite development by Taylor's Theorem, where $V'$ denotes what $V$ becomes when we change $y'$ into $y'+\delta y'$.

**8.** We may now proceed to a full solution of the problem. We have

$$V = \sqrt{1 + y'^2},\ \frac{dV}{dy'} = \frac{y'}{\sqrt{1 + y'^2}} = \frac{y'}{V},$$

$$\frac{d^2V}{dy'^2} = \frac{1}{\sqrt{(1 + y'^2)^3}} = \frac{1}{V^3},\ \frac{d^3V}{dy'^3} = -\frac{3y'}{\sqrt{(1 + y'^2)^5}} = -\frac{3y'}{V^5}.$$

Hence, as these and the succeeding partial differential coefficients of $V$ with respect to $y'$ are all finite, we can develop $l'$ by Taylor's Theorem, and equation (3) gives

$$\delta l = \int_{x_0}^{x_1} \left\{ \frac{y'}{V} \delta y' + \frac{1}{2V^3} \delta y'^2 - \frac{y'}{2V^5} \delta y'^3 + \text{etc.} \right\} dx, \tag{5}$$

in which we have first to consider the expression

$$\int_{x_0}^{x_1} \frac{y'}{V} \delta y' dx = 0. \tag{6}$$

This equation is of course satisfied by making $\frac{y'}{V}$ zero, which gives necessarily $y'$ zero, and $y$ a constant. This would make the required curve a right line, coinciding with, or parallel to, the axis of $x$. While this solution is correct so far as the general form of the required curve is concerned, it will not be always possible to draw such a line through two fixed points given at pleasure, unless we are at liberty to assume the axis of $x$ so as to make $y_1$ and $y_0$ equal, which is not contemplated. We must, therefore, seek another solution.

**9.** We will begin by transforming $\delta y'$ thus;

$$y' = \frac{dy}{dx}.$$

Change $y$ into $y + \delta y$, while $x$, and consequently $dx$, undergo

no alteration. Then denoting the new value of $y'$ by $Y'$, we have

$$Y' = \frac{d}{dx}(y + \delta y) = \frac{dy}{dx} + \frac{d\delta y}{dx}.$$

Whence, subtracting from the first member $y'$, and from the last its equal $\frac{dy}{dx}$, we have

$$Y' - y' = \frac{d\delta y}{dx}.$$

But $Y' - y' = \delta y'$, whence

$$\delta y' = \frac{d\delta y}{dx}. \tag{A}$$

In like manner,

$$y'' = \frac{d^2 y}{dx^2}.$$

Change $y$ into $y + \delta y$. Then

$$Y'' = \frac{d^2}{dx^2}(y + \delta y) = \frac{d^2 y}{dx^2} + \frac{d^2 \delta y}{dx^2}.$$

$$Y'' - y'' = \delta y'' = \frac{d^2 \delta y}{dx^2}; \tag{B}$$

and, similarly,

$$\delta y^{(n)} = \frac{d^n \delta y}{dx^n}, \tag{C}$$

where $n$ is any positive integer.

**10.** Equation (6) may now be written

$$\int_{x_0}^{x_1} \frac{y'}{V} \frac{d\delta y}{dx} dx = 0. \tag{7}$$

But integrating by parts, we have

$$\int \frac{y'}{V}\frac{d\delta y}{dx}\,dx = \frac{y'}{V}\,\delta y - \int \frac{d}{dx}\frac{y'}{V}\,.\,\delta y\;dx,$$

$$\int_{x_0}^{x_1}\frac{y'}{V}\frac{d\delta y}{dx}\,dx = \left(\frac{y'}{V}\,\delta y\right)_1 - \left(\frac{y'}{V}\,\delta y\right)_0 - \int_{x_0}^{x_1}\frac{d}{dx}\frac{y'}{V}\,.\,\delta y\,dx = 0; \tag{8}$$

where the suffix 1 denotes what the quantities affected become when $x$ is $x_1$, and 0 what the same quantities become when $x$ is $x_0$. But since the two points through which the required line must pass are fixed, $\delta y_1$ and $\delta y_0$ are each zero; that is, $y$ receives no increment at these points, and therefore (8) becomes

$$-\int_{x_0}^{x_1}\frac{d}{dx}\frac{y'}{V}\,.\,\delta y\;dx = 0. \tag{9}$$

This equation can be satisfied by writing

$$\frac{d}{dx}\frac{y'}{V} = 0, \qquad \frac{y'}{V} = c.$$

Squaring, clearing fractions, and transposing, we have

$$y'^2 - c^2y'^2 = c^2, \qquad y' = \frac{\pm c}{\sqrt{1 - c^2}} = a, \qquad y = ax + b,$$

the general equation of the straight line.

**11.** It will be seen that the solution $y' = 0$ is only a particular case of the more general one just obtained, and we are therefore led to inquire why the method pursued in Art. 8 did not give a satisfactory result. Now, since we have the equations $\delta y' = \frac{d\delta y}{dx}$, $\frac{y'}{V} = c$, (6) may be written

$$\int_{x_0}^{x_1}\frac{y'}{V}\,\delta y'\;dx = \int_{x_0}^{x_1} c\,d\delta y = 0,$$

whence, by integration,

$$c(\delta y_1 - \delta y_0) = 0;$$

and because both $\delta y_1$ and $\delta y_0$ are zero, this equation can be satisfied without making $c$ zero.

The error, therefore, in Art. 8 appears to have arisen from the fact that we required the curve to pass through two fixed points, and then entirely disregarded that condition in obtaining our solution. But (9) was established by expressly imposing this condition upon the problem; and as there are no further conditions to be imposed, and as $\delta y$ cannot be further transformed, that equation can only be satisfied by equating to zero the coefficient of $\delta y\, dx$ in that equation.

**12.** Resuming equation (5), let us next consider the term of the second order,

$$\int_{x_0}^{x_1} \frac{1}{2V^3} \delta y'^2 dx. \qquad (10)$$

If the solution given above be a true minimum, this term must become positive, or must reduce to zero. Now since $x$ is the independent variable, $dx$ is always supposed to be estimated positively; and as $\delta y'^2$ can never be negative, if we also regard $V$ as positive, we see that every element of (10) is positive, and that consequently the integral itself must be of the same sign. We conclude, therefore, that a right line is the plain curve of minimum length between two fixed points.

If the coefficient of $\delta y'^2 dx$ in (10), which we may call $Z$, could have changed its sign within the given limits of integration—that is, if $Z$ could have been positive throughout some portions of the curve, and negative throughout others—we could make (10) take either sign, and there could be neither a maximum nor a minimum. For by varying $y'$ throughout those portions of the curve for which $Z$ was negative, while leaving the other portions unvaried, the integral would become negative, or by pursuing an opposite

course it would become positive. Hence, in this and similar cases, the coefficient of $\delta y'^2 dx$ must be of invariable sign for all values of $x$ from $x_0$ to $x_1$.

If $Z$ could have reduced to zero throughout the whole range of integration, thus rendering the integral itself zero, we might generally infer that the solution was neither a maximum nor a minimum. For in order to the existence of either, the term of the third order involving $\delta y'^3$ must also vanish, which would seldom if ever occur.

It will be observed that the term of the second order is positive whether the extremities of the required curve are supposed to be fixed or not. But if we disregard this condition, the terms of the first order would not vanish, so that we would not obtain a minimum, except, indeed, we adopt the particular solution of Art. 8. We shall, however, subsequently show that when the limiting values of $x$ only are given—that is, when the required curve is merely to have its extremities upon two fixed lines perpendicular to the axis of $x$—the solution of Art. 8 is that which must be taken.

**13.** In the preceding discussion we have merely proved that the straight line between two fixed points is shorter than any other plane curve which could be derived from it by making indefinitely small changes in the inclination of its tangents to the axis of $x$, either in certain portions or throughout its whole extent. We could not, therefore, by the use of the calculus of variations alone, become certain that the straight line is the shortest plane curve which can be drawn between two fixed points, but merely that it is a curve of minimum length, the existence of other minima being possible; one of which might, perhaps, be less than the present, and might itself be the shortest curve.

Again, the preceding method does not permit us to compare the straight line with all other plane curves which can be drawn indefinitely close to it. For in developing $I'$, Art. 3,

we were obliged to ascribe indefinitely small increments or variations to $y'$ only, since $y$ did not directly or explicitly occur in $I$. Hence the curve which we derive by variations can have no abrupt change of direction; because no such change could occur without rendering $\delta y'$ appreciably large at that point. Therefore all curves with cusps, and all systems of broken lines, are excluded from the comparison, although it is evident from the figure that such curves might be drawn without making the variations of $y$ appreciable, but only those of $y'$.

**14.** From the remarks of the preceding article, which were deemed necessary in order to guard the reader against certain misconceptions which are common among students of this subject, it must not be inferred that the calculus of variations is of little use as a method of solving questions of maxima and minima. For we shall see as we advance that it can in general be made to give a satisfactory solution when such a solution exists. Indeed, the recent discoveries relative to the theory of discontinuity, which are due chiefly to the labors of Prof. Todhunter, and of which we shall speak hereafter, show that this branch of the calculus does not in reality fail to present solutions even in very many of those cases in which its failure has been hitherto assumed.

**15.** It remains only to determine the constants $a$ and $b$ which occur in the general solution. It will appear that since the required line is to pass through two fixed points whose co-ordinates are $x_0$, $y_0$, $x_1$, $y_1$, we must have

$$a = \frac{y_1 - y_0}{x_1 - x_0};$$

and therefore so soon as these quantities are given $a$ becomes known. Then to determine $b$, we have $y_0 = ax_0 + b$,

$$b = y_0 - ax_0 = y_0 - \frac{y_1 - y_0}{x_1 - x_0} x_0,$$

and thus $b$ is also known when $x_0$, $x_1$, $y_0$, $y_1$, are fully given.

**16.** In further illustration of our subject we next proceed to consider another problem, the solution of which is not so generally known.

### Problem II.

*It is required to determine the equation of the plane curve, down which a particle, acted upon by gravity alone, would descend from one fixed point to another in the shortest possible time.*

Let $a$ be the upper and $b$ the lower point. Assume the axis of $x$ vertically downward, and $a$ as the origin of co-ordinates. Also let the variable $s$ be the length of the required curve at any point measured from $a$; $v$, the velocity of the particle at the same point; and $t$, its time of descent from $a$ to that point. Then we wish to determine the curve which will render $T$ a minimum, where $T$ is the total time of descent from $a$ to $b$, or what $t$ becomes at the point $b$. We must first then find $t$ as a function of $x$ and $y$, or their differentials.

Now, from the well-known differential equations of motion in mechanics, we have

$$dt = \frac{ds}{v}, \qquad (1)$$

and, Art. 2,

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\, dx.$$

We also know that the particle loses no velocity in passing from one point to another of a curve with no abrupt change of direction and that [illegible] it start from a state of rest at [illegible] its velocity at any point of the curve must equal that which it would have acquired in falling freely through the same vertical [illegible]. Hence we shall have

[illegible]

[illegible] being the acceleration due to gravity. Therefore [illegible]

[illegible]

and

[illegible]

[illegible] minimum.

[illegible] member [illegible]

[illegible]

[illegible] constant [illegible] minimum [illegible]

[illegible]

[illegible]

$$\delta V = \frac{dV}{dy'}\delta y' + \frac{1}{2}\frac{d^2V}{dy'^2}\delta y'^2 + \text{etc.}$$

$$\delta U = \int_{x_0}^{x_1}\left\{\frac{dV}{dy'}\delta y' + \frac{1}{2}\frac{d^2V}{dy'^2}\delta y'^2 + \text{etc.}\right\}dx$$

$$= \int_{x_0}^{x_1}\frac{y'}{\sqrt{x(1+y'^2)}}\delta y' dx + \int_{x_0}^{x_1}\frac{1}{2\sqrt{x(1+y'^2)^3}}\delta y'^2 dx + \text{etc.} \quad (4)$$

We shall not in future develop any variation beyond the terms of the second order, since if the terms of the first two orders should become zero, there could rarely if ever be either a maximum or a minimum, as explained in Art. 12.

Hence we must have

$$\int_{x_0}^{x_1}\frac{y'}{\sqrt{x(1+y'^2)}}\delta y' dx = 0. \quad (5)$$

But since the two extreme points are fixed, we must impose this condition upon the problem by integrating (5) by parts, as in the preceding problem, and neglecting the terms thus freed from the integral sign, because containing $\delta y_0$ and $\delta y_1$. Performing this operation, we shall obtain

$$-\int_{x_0}^{x_1}\frac{d}{dx}\frac{y'}{\sqrt{x(1+y'^2)}}\cdot\delta y\, dx = 0, \quad (6)$$

$$\frac{d}{dx}\frac{y'}{\sqrt{x(1+y'^2)}} = 0, \quad (7)$$

$$\frac{y'}{\sqrt{x(1+y'^2)}} = c. \quad (8)$$

Now since $c$ in the last equation is an arbitrary constant, make

it equal to $\frac{1}{\sqrt{a}}$. Then squaring, clearing fractions, and transposing, we have

$$y'^2 - \frac{xy'^2}{a} = \frac{x}{a}.$$

Whence solving for $y'$, we obtain

$$y' = \frac{\sqrt{x}}{\sqrt{a - x}}, \tag{9}$$

which is known to be the differential equation of the cycloid. Therefore

$$y = \text{versin}^{-1} x - \sqrt{ax - x^2} + b, \tag{10}$$

where $a$ is twice the radius of the generating circle, and $b$ is zero, because the origin was taken at the upper point. The last equation may be finally written thus:

$$y = r \,\text{versin}^{-1} \frac{x}{r} - \sqrt{2rx - x^2}, \tag{11}$$

where the circular function is natural, and $r$ is the radius of the generating circle.

**18.** By disregarding the condition that the curve must pass through the two fixed points, we shall, as in the preceding problem, obtain from (5),

$$\frac{y'}{\sqrt{x(1 + y'^2)}} = 0,$$

which makes $y'$ zero, and $y$ a constant, which must also be zero, because the curve passes through the origin. Therefore the curve would in this case coincide throughout with

the axis of $x$, which solution could only be possible when the two points were in the same vertical line, and then its truth is self-evident.

**19.** Let us now consider the term of the second order, viz.,

$$\int_{x_0}^{x_1} \frac{1}{2\sqrt{x(1+y'^2)^3}}\,\delta y'^2 dx. \tag{12}$$

If the cycloid be the true solution of our problem, this term must become positive, whether $y'$ be varied throughout the whole integral or only throughout certain portions taken at pleasure. To satisfy this condition it is merely necessary that $Z$, the coefficient of $\delta y'^2 dx$ in (12), shall become positive and not change its sign as we pass from $a$ to $b$. But since $x$ cannot become negative in this problem, the square root of $x$ is real and may be considered as always positive from $a$ to $b$; then, as we may regard $\sqrt{1+y'^2}$ as always positive, the above conditions are satisfied, and we conclude that the cycloid, having a cusp at $a$, its base horizontal, and its vertex downward, is a solution of our problem.

Let us also try the solution $y' = 0$ of Art. 18; this will reduce (12) to

$$\int_{x_0}^{x_1} \frac{1}{2\sqrt{x}}\,\delta y'^2 dx,$$

which will also become necessarily positive if we assume $\sqrt{x}$ to be positive. Thus this solution likewise, when it is possible, renders $T$ a minimum, as it evidently should.

**20.** Remarks similar to those made in Art. 13 apply also to this example. For it is plain that we have only compared the cycloid as a curve of descent, with all other curves passing through the given points, having no abrupt change of direction, and drawn indefinitely near to it. Hence we have

in reality only shown that the cycloid is one of the curves which renders $T$ a minimum, the term minimum being used in the technical sense hitherto explained. However, as in the former problem, these restrictions are merely theoretical, and are noticed in order to prevent misconceptions which might occasion difficulty in subsequent discussions.

For in the present case the cycloid between two points is undoubtedly the curve of quickest descent from one to the other, and from this property it is often called the brachistochrone.

**21.** In addition to what has been already said, we must here call attention to another point which is often passed over by elementary writers on this subject. Suppose $y'$ to become infinite for some point within the range of integration, as it does at the vertex of the cycloid. Then when we change $y'$ into $y' + \delta y'$, if we regard, as we must by the theory of the subject, $\delta y'$ as taken arbitrarily, but always indefinitely small, we can make the new or derived curve assume any form we please, except that its tangent at $X$ must have the same direction as that of the cycloid at the vertex, where $X$ is the abscissa of the vertex. For suppose the vertex tangent of the cycloid to undergo a slight change of direction, so that its new angle of inclination to $x$ may differ from a right angle in an indefinitely small degree. Then we cannot assert that this small change of direction could be produced by an indefinitely small change in the value of $y'$, or the natural tangent of the right angle. That is, owing to the indeterminate nature of infinity, we cannot with certainty apply the method of variations to any element of the integral which is affected by an infinite value of $y'$, and hence the integral must not be extended so as to include this element. In the present case, then, we are only sure of a minimum so long as we are not obliged to go beyond the vertex of the cycloid for $b$.

But the occurrence of an infinite value of $y'$ in any case

will not warrant us in concluding that the solution does not give a true maximum or minimum, even when the integral includes that value of $y'$. All that we can say is that the proposed method becomes inapplicable. Indeed, we shall have occasion to show that sometimes, by changing to polar coordinates, or by some other change of the independent variable, the integral may in these cases be freed from infinite quantities, and the previous solution shown to give a true maximum or minimum.

Of course if we regard $\delta y'$ as zero when $y'$ becomes infinite—that is, consider the tangent to the curve as fixed at that point—the variation of the element becoming zero, may be included in the development, and all difficulty disappears.

It will be observed that $V$ becomes infinite at $A$, and the solution is therefore still subject to any objection, but there would seem to be none, which can arise from this fact.

## CASE 2.

**22.** As a means of still further extending our knowledge of variations, let us resume the preceding problem, merely taking the horizontal as the axis of $x$.

Then, the notation and the other conditions being unchanged, we must, as before, render $T$ a minimum. But, as formerly,

$$dt = \frac{ds}{v}, \qquad ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx,$$

where $y'$ now means the natural tangent of the angle which any tangent to the curve makes with the horizontal instead of the vertical axis. Also, $v = \sqrt{2gy}$, so that, neglecting, as be- before, the constant factor, we must minimize the expression

$$U = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y}}\,dx = \int_{x_0}^{x_1} V dx.$$

Now in $V$ change $y$ into $y+\delta y$, and $y'$ into $y'+\delta y'$. Then we may develop $V'$, or the new state of $V$, by the extension of Taylor's Theorem, thus:

$$V' = V + \frac{dV}{dy}\delta y + \frac{dV}{dy'}\delta y'$$
$$+ \frac{1}{2}\left(\frac{d^2V}{dy^2}\delta y^2 + \frac{2d^2V}{dydy'}\delta y\delta y' + \frac{d^2V}{dy'^2}\delta y'^2\right) + \text{etc.}$$

We also have

$$U' = \int_{x_0}^{x_1} V'dx,$$

where $U'$ is what $U$ becomes when we change $V$ into $V+\delta V$ or into $V'$, $dx$ being unaltered. Hence calling $U'-U$, $\delta U$, we have

$$\delta U = \int_{x_0}^{x_1}\left\{\frac{dV}{dy}\delta y + \frac{dV}{dy'}\delta y'\right\}dx$$
$$+ \frac{1}{2}\int_{x_0}^{x_1}\left\{\frac{d^2V}{dy^2}\delta y^2 + \frac{2d^2V}{dydy'}\delta y\,\delta y' + \frac{d^2V}{dx'^2}\delta y'^2\right\}dx + \text{etc.} \quad (1)$$

Indeed, it is evident that a similar course could be pursued should $V$ contain any number of quantities capable of being varied.

**23.** It may be well before proceeding further to refer briefly to the subject of maxima and minima of functions involving more than one variable, as it is developed by the differential calculus.

Let $f$ be a function of $x, y, z$, etc. Give small increments, $h$, $i$, $k$, etc., to $x$, $y$, $z$, respectively, and develop $f'$, the new state of $f$, by Taylor's Theorem. Then the terms of the first order in $f'-f$ will be

$$\frac{df}{dx}h + \frac{df}{dy}i + \frac{df}{dz}k + \text{etc.},$$

which must collectively vanish; and if the quantities $h$, $i$, $k$, etc., be independent, each of the partial differential coefficients of $f$ must also vanish. Then the terms of the second order,

$$\frac{1}{2}\left(\frac{d^2f}{dx^2}h^2 + 2\frac{d^2f}{dxdy}hi + \frac{d^2f}{dy^2}i^2 + \text{etc.}\right),$$

must become collectively negative for a maximum and positive for a minimum. Also, if the increments be independent, the second partial differential coefficients of $f$ must fulfil certain conditions among themselves, for an account of which, as they have no application here, the reader is referred to works on the differential calculus.

**24.** The expression for $\delta U$ in (1) is similar to that for $f'-f$, only each term is multiplied by $dx$, and is under an integral sign, $\delta y$ and $\delta y'$ taking the place of $h$ and $i$, $dx$ being regarded as constant. In the present case, therefore, the two integrals of the first order in (1) must collectively vanish, while the three integrals of the second order must become collectively positive.

**25.** We have

$$\frac{dV}{dy} = -\frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}}, \quad \frac{dV}{dy'} = \frac{y'}{\sqrt{(1+y'^2)y}}, \quad \frac{d^2V}{dy^2} = \frac{3\sqrt{1+y'^2}}{4y^{\frac{5}{2}}},$$

$$\frac{d^2V}{dy\,dy'} = -\frac{y'}{2\sqrt{(1+y'^2)y^3}}, \quad \frac{d^2V}{dy'^2} = \frac{1}{\sqrt{(1+y'^2)^3y}}.$$

Hence equation (1) becomes

$$\delta U = \int_{x_0}^{x_1}\left\{-\frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}}\delta y + \frac{y'}{\sqrt{(1+y'^2)y}}\delta y'\right\}dx$$

$$+\frac{1}{2}\int_{x_0}^{x_1}\left\{\frac{3\sqrt{1+y'^2}}{4y^{\frac{5}{2}}}\delta y^2 - \frac{y'}{\sqrt{(1+y'^2)y^3}}\delta y\delta y'\right.$$

$$\left. + \frac{1}{\sqrt{(1+y'^2)^3y}}\delta y'^2\right\}dx. \qquad (2)$$

Whence we have

$$\int_{x_0}^{x_1}\left\{-\frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}}\,\delta y+\frac{y'}{\sqrt{(1+y'^2)y}}\,\delta y'\right\}dx=0. \tag{3}$$

Now, it might at first appear that we could regard $\delta y$ and $\delta y'$ as independent, and thus might equate to zero each of the integrals in (3). But since the curve is to pass through two fixed points, this condition, which has not yet been regarded, must be imposed upon the problem, and may be said to limit, in some sense, the independence of $\delta y$ and $\delta y'$. This condition can be imposed by means of the second integral only, since the first is incapable of any further integration. For putting for $\delta y'$ its value from (A), we have

$$\int\frac{y'}{\sqrt{(1+y'^2)y}}\,\frac{d\delta y}{dx}\,dx=\frac{y'}{\sqrt{(1+y'^2)y}}\,\delta y-\int\frac{d}{dx}\,\frac{y'}{\sqrt{(1+y'^2)y}}.\delta y\,dx.$$

Hence, since $\delta y_1$ and $\delta y_0$ are zero, when we make the integral definite, the two terms which will be without the sign of integration will disappear, and we shall have

$$\int_{x_0}^{x_1}\frac{y'}{\sqrt{(1+y'^2)y}}\,\delta y'dx=-\int_{x_0}^{x_1}\frac{d}{dx}\,\frac{y'}{\sqrt{(1+y'^2)y}}\cdot\delta y\,dx,$$

and therefore (3) may be written

$$-\int_{x_0}^{x_1}\left\{\frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}}+\frac{d}{dx}\,\frac{y'}{\sqrt{(1+y'^2)y}}\right\}\delta y\,dx=0. \tag{4}$$

Thus $\delta y'$ has been eliminated, and there being no further conditions to impose, (4) can only be satisfied by writing

$$\frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}}+\frac{d}{dx}\,\frac{y'}{\sqrt{(1+y'^2)y}}=0. \tag{5}$$

Multiply the first term by $dy$, and the second by its equal, $y'dx$, and we have

$$\frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}}dy + y'\frac{d}{dx}\frac{y'}{\sqrt{(1+y'^2)y}}dx = 0. \tag{6}$$

Then by parts,

$$\int\frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}}dy = -\frac{\sqrt{1+y'^2}}{y^{\frac{1}{2}}} + \int\frac{y'}{\sqrt{(1+y'^2)y}}dy';$$

and again by parts,

$$\int\frac{y'}{\sqrt{(1+y'^2)y}}dy' = \frac{y'^2}{\sqrt{(1+y'^2)y}} - \int y'\frac{d}{dx}\frac{y'}{\sqrt{(1+y'^2)y}}dx;$$

so that we have, finally,

$$\frac{y'^2}{\sqrt{(1+y'^2)y}} - \frac{\sqrt{1+y'^2}}{y^{\frac{1}{2}}} = \text{a constant, say } \frac{-1}{\sqrt{a}}. \tag{7}$$

Now reducing the first member to a common denominator, we have

$$-\frac{1}{\sqrt{(1+y'^2)y}} = \frac{-1}{\sqrt{a}}, \quad y(1+y'^2) = a, \quad y'^2 = \frac{a-y}{y}; \tag{8}$$

which last equation cannot be integrated by solving for $y'$. But we readily obtain

$$\frac{1}{y'} \text{ or } \frac{dx}{dy} = \frac{\sqrt{y}}{\sqrt{a-y}},$$

which is as before the differential equation of the cycloid, in which $a$ equals $2r$; only $x$ and $y$ have been interchanged, as will appear from equation (10), Art. 17.

**26.** If we disregard the condition that the curve is to pass through two fixed points, we shall have, from (2),

$$\int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}}\,\delta y\,dx = 0, \qquad \int_{x_0}^{x_1} \frac{y'}{\sqrt{(1+y'^2)y}}\,\delta y'dx = 0. \qquad (9)$$

Now the first of these equations can only be satisfied by equating to zero the coefficient of $\delta y dx$, and then, as we may evidently neglect the supposition that $y$ is infinite throughout the curve, we have, necessarily,

$$\sqrt{1+y'^2} = 0, \qquad y' = \pm\sqrt{-1}\,;$$

a result which shows that a solution by this method is impossible.

The solution $y' = 0$ of Art. 18, which will become in this case $y' = \infty$, is also suggested by this method; for if in the second of equations (8) we make $a$ infinite, then, since $y$ cannot be always infinite, we shall find that $y'$ is infinite. This solution, representing the vertical through $A$, has been already shown to give a true minimum; although the considerations of Art. 20 show that it could not be investigated so long as the horizontal is taken as the independent variable. This case then exemplifies the remarks there made relative to overcoming, by a change of the independent variable, the difficulty presented by the occurrence of infinite quantities.

**27.** Let us now examine the sign of the terms of the second order in $\delta U$. Since those of the first order vanish, we have, from (2),

$$\delta U = \int_{x_0}^{x_1} \left\{ \frac{3\sqrt{1+y'^2}}{8y^{\frac{5}{2}}}\,\delta y^2 - \frac{y'}{2\sqrt{1+y'^2}\,y^{\frac{3}{2}}}\,\delta y\delta y' + \frac{\delta y'^2}{2\sqrt{(1+y'^2)^3 y}} \right\} dx. \qquad (10)$$

From the second of equations (8) we have

$$\sqrt{y(1+y'^2)} = \sqrt{a},$$

and therefore (10) becomes

$$\delta U = \int_{x_0}^{x_1} \left\{ \frac{3\sqrt{a}}{8y^{\frac{5}{2}}} \delta y^2 - \frac{y'}{2y\sqrt{a}} \delta y\, \delta y' + \frac{y}{2a\sqrt{a}} \delta y'^2 \right\} dx. \quad (11)$$

But $\frac{3\sqrt{a}}{8y^{\frac{5}{2}}}$ can be written $\frac{3r}{2\sqrt{a}\,.\,2y^{\frac{5}{2}}}$, where $r = \frac{a}{2}$, or the radius of the generating circle. Whence (11) becomes

$$\delta U = \frac{1}{2\sqrt{a}} \int_{x_0}^{x_1} \left\{ \frac{3r}{2y^{\frac{5}{2}}} \delta y^2 - \frac{y'}{y} \delta y\, \delta y' + \frac{y}{a} \delta y'^2 \right\} dx. \quad (12)$$

But, from equation (A),

$$\int \frac{y'}{y} \delta y\, \delta y' dx = \frac{y'}{y} \frac{\delta y^2}{2} - \int \frac{\delta y^2}{2} \frac{d}{dx} \frac{y'}{y} \,.\, dx. \quad (13)$$

Put $l$ for $\frac{y'}{y}$. Then

$$\int_{x_0}^{x_1} l\, \delta y\, \delta y' dx = \frac{1}{2} \left( l_1\, \delta y_1^2 - l_0\, \delta y_0^2 \right) - \frac{1}{2} \int_{x_0}^{x_1} \delta y^2 \frac{dl}{dx} dx. \quad (14)$$

But since the extreme points of the curve are fixed, $\delta y_1$ and $\delta y_0$ are each zero, and we have

$$\int_{x_0}^{x_1} l\, \delta y\, \delta y' dx = -\frac{1}{2} \int_{x_0}^{x_1} \delta y^2 \frac{dl}{dx} dx. \quad (15)$$

But

$$\frac{dl}{dx} dx = dl = \frac{y dy' - y' dy}{y^2} = \frac{dy'}{y} - \frac{y'^2}{y^2} dx;$$

and because $dy = y' dx$, the last equation may be written

$$\frac{dl}{dx} dx = \left\{ \frac{1}{y} \frac{y' dy'}{dy} - \frac{y'^2}{y^2} \right\} dx. \quad (16)$$

Now differentiating the third of equations (8) and dividing by 2, we have

$$y'dy' = -\frac{ady}{2y^2} = -\frac{rdy}{y^2},$$

and therefore (16) becomes

$$\frac{dl}{dx}dx = -\left\{\frac{r}{y^2} + \frac{y'^2}{y^2}\right\}dx.$$

But, from the third of equations (8),

$$y'^2 = \frac{2r - y}{y},$$

whence

$$\frac{dl}{dx}dx = -\frac{3r - y}{y^3}dx. \tag{17}$$

Therefore

$$\int_{x_0}^{x_1}\frac{y'}{y}\,\delta y\,\delta y'dx = \frac{1}{2}\int_{x_0}^{x_1}\delta y^2\,\frac{3r - y}{y^3}\,dx. \tag{18}$$

Substituting this value in (12), we have, finally,

$$\delta U = \frac{1}{2\sqrt{a}}\int_{x_0}^{x_1}\left\{\frac{\delta y^2}{2y^2} + \frac{y}{a}\,\delta y'^2\right\}dx, \tag{19}$$

and it is evident that this integral is positive, since each of its elements is positive.

**28.** Although we might infer from the preceding article that we have a minimum, that term being used in its technical sense, still our investigation will not be entirely trustworthy unless we regard the direction of the tangent at $A$ as absolutely fixed. For we have seen that $y'^2 = \frac{2r - y}{y}$, and therefore when

$y$ is zero, $y'$ becomes infinite. That is, we cannot with confidence include in our investigation every element of the definite integral $U$, because at $A$, $V$ becomes infinite. We cannot, however, conclude that there is not a minimum, because we do not know what effect a variation of $y'$ in this element would have upon the general result. Indeed, we do know that if the second point be not beyond the vertex, we have a true minimum, and we now see also that if the tangent at $A$ be fixed—that is, if the cycloid be compared with any other derived curve whose tangent is at right angles to the horizontal—we shall in any case have a minimum.

The term derived will be used to denote any curve which can be obtained from the original or primitive curve by the method of variations, and must therefore be always indefinitely near to its primitive, and without abrupt change of direction.

**29.** The preceding discussion shows the advantage of taking the vertical as the independent variable. For while the result by either method is the same, as indeed it must be in every case, it is much more easily obtained by the former. This is due to the fact that in the former case $x$, being incapable of variation, enters the function $V$, thus leaving $y'$ only to be varied, while in the latter $y$ and $y'$, both being capable of variation, enter $V$, thus rendering the problem one of two variables.

When we come to the terms of the second order, the results apparently agree also. But while that in the former case is readily obtained, and is probably entirely trustworthy so long as we do not wish to pass the vertex, in the latter case some transformation is required in order to obtain any result, and even then, owing to the occurrence of an infinite value of $y'$ at the outset, we cannot rely implicitly upon our investigation unless we regard the derived curve as having at $A$ the same tangent as its primitive; that is, the vertical.

## Problem III.

**30.** *It is required to determine the form of the plane curve which shall pass through two fixed points, and which shall include between itself, its evolute, and its radii of curvature at the two fixed points a minimum area; the extreme tangents of the required curve being also fixed.*

As before, let $ds$ be an element of the required curve, $r$ the radius of curvature, and $U$ the area which is to become a minimum. Then

$$U = \int_{s_0}^{s_1} r\,ds, \tag{1}$$

and we must first express $U$ in terms of $x, y, y'$, etc.
We have

$$ds = \sqrt{1 + y'^2}\,dx,$$

$$r = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,d^2y} = \frac{ds^3}{dx\,d^2y} = \frac{ds^3}{y''dx^3} = \frac{\sqrt{(1 + y'^2)^3}}{y''}, \tag{2}$$

the sign $\pm$ having been disregarded. Substituting these values, and assuming that the curve is to be concave to the axis of $x$, and $y''$ therefore negative, (1) may be written

$$U = -\int_{x_0}^{x_1} \frac{(1 + y'^2)^2}{y''}\,dx = \int_{x_0}^{x_1} V dx. \tag{3}$$

Now change $y'$ into $y' + \delta y'$, $y''$ into $y'' + \delta y''$, and develop as before. Then including the terms of the second order, we have

$$\delta U = -\int_{x_0}^{x_1} \left\{ \frac{4y'(1 + y'^2)}{y''}\delta y' - \frac{(1 + y'^2)^2}{y''^2}\delta y'' \right\} dx$$

$$-\frac{1}{2}\int_{x_0}^{x_1} \left\{ \frac{4(1 + 3y'^2)}{y''}\delta y'^2 - \frac{8y'(1 + y'^2)}{y''^2}\delta y'\delta y'' \right.$$

$$\left. + \frac{2(1 + y'^2)^2}{y''^3}\delta y''^2 \right\} dx. \tag{4}$$

Whence we must have

$$\int_{x_0}^{x_1}\left\{\frac{4y'(1+y'^2)}{y''}\,\delta y' - \frac{(1+y'^2)^2}{y''^2}\,\delta y''\right\}dx = 0. \tag{5}$$

Now it is plain, as before, that the two integrals combined in the last equation are not independent, there being here two conditions to be imposed upon the problem; namely, that $\delta y_1$ and $\delta y_0$ shall vanish, and also that $\delta y_1'$ and $\delta y_0'$ shall vanish.

To impose these conditions, we have only to extend the method already employed. Thus, putting $K$ for $\dfrac{4y'(1+y'^2)}{y''}$, we have

$$\int K\,\delta y'dx = K\,\delta y - \int\frac{dK}{dx}\,\delta y\,dx,$$

$$\int_{x_0}^{x_1} K\,\delta y'dx = K_1\,\delta y_1 - K_0\,\delta y_0 - \int_{x_0}^{x_1}\frac{dK}{dx}\,\delta y\,dx. \tag{6}$$

Also putting $L$ for $\dfrac{(1+y'^2)^2}{y''^2}$, and observing that

$$\delta y'' = \frac{d^2\,\delta y}{dx^2} = \frac{d\,\delta y'}{dx},$$

we have

$$-\int_{x_0}^{x_1} L\,\delta y''dx = -(L_1\,\delta y_1' - L_0\,\delta y_0') + \int_{x_0}^{x_1}\frac{dL}{dx}\,\delta y'dx.$$

And in a similar manner we obtain

$$\int_{x_0}^{x_1}\frac{dL}{dx}\,\delta y'dx = \left(\frac{dL}{dx}\right)_1\delta y_1 - \left(\frac{dL}{dx}\right)_0\delta y_0 - \int_{x_0}^{x_1}\frac{d^2L}{dx^2}\,\delta y\,dx,$$

$$-\int_{x_0}^{x_1} L\,\delta y''dx = \left(\frac{dL}{dx}\,\delta y - L\,\delta y'\right)_1 - \left(\frac{dL}{dx}\,\delta y - L\,\delta y'\right)_0$$

$$-\int_{x_0}^{x_1}\frac{d^2L}{dx^2}\,\delta y\,dx. \tag{7}$$

Collecting and arranging these results, (5) becomes

$$\int_{x_0}^{x_1} \left\{ \frac{4y'(1+y'^2)}{y''}\,\delta y' - \frac{(1+y'^2)^2}{y''^2}\,\delta y'' \right\} dx =$$

$$\left\{ \left(K+\frac{dL}{dx}\right)\delta y - L\,\delta y' \right\}_1 - \left\{ \left(K+\frac{dL}{dx}\right)\delta y - L\,dy' \right\}_0$$

$$-\int_{x_0}^{x_1} \left\{ \frac{dK}{dx} + \frac{d^2L}{dx^2} \right\} \delta y\,dx = 0. \tag{8}$$

Now, if we suppose $\delta y_1$, $\delta y_1'$, $\delta y_0$, $\delta y_0'$, to severally vanish, we shall thereby impose the two given conditions upon the problem, and (8) will become

$$-\int_{x_0}^{x_1} \left\{ \frac{dK}{dx} + \frac{d^2L}{dx^2} \right\} \delta y\,dx = 0. \tag{9}$$

As there are no further conditions to impose, this equation can only be satisfied by writing

$$\frac{dK}{dx} + \frac{d^2L}{dx^2} = 0. \tag{10}$$

Restoring the values of $K$ and $L$, and integrating, we have

$$\frac{4y'(1+y'^2)}{y''} + \frac{d}{dx}\frac{(1+y'^2)^2}{y''^2} + c = 0, \tag{11}$$

which, since $dy' = y''dx$, may be written

$$\frac{4y'(1+y'^2)}{y''}dy' + y''d\,.\,\frac{(1+y'^2)^2}{y''^2} + c\,dy' = 0. \tag{12}$$

Then integrating by parts, we have

$$\int \frac{4y'(1+y'^2)}{y''}\,dy' = \frac{(1+y'^2)^2}{y''} + \int \frac{(1+y'^2)^2}{y''^2}\,dy'',$$

$$\int \frac{(1+y'^2)^2}{y''^2}\,dy'' = \frac{(1+y'^2)^2}{y''^2}\,y'' - \int y''d\,.\,\frac{(1+y'^2)^2}{y''^2}.$$

Hence (12) gives

$$\frac{2(1+y'^2)^2}{y''}+cy'+c'=0, \tag{13}$$

$$\frac{(cy'+c')y''}{(1+y'^2)^2}=-2. \tag{14}$$

But from equations (1), (2) and (3) we readily obtain

$$\frac{y''}{(1+y'^2)^2}=\frac{dx}{-r\,ds},$$

and substituting this value in (14), observing that $y'dx=dy$, we have

$$c\frac{dy}{ds}+c'\frac{dx}{ds}=2r. \tag{15}$$

Let $t$ denote the angle which the tangent to the required curve at any point makes with the axis of $x$. Then $\frac{dy}{ds}=\sin t$, and $\frac{dx}{ds}=\cos t$. Also, let $b$ be the constant angle whose natural tangent is $\frac{c}{c'}$. Then $c=h\sin b$, and $c'=h\cos b$; $h$ being some constant at present unknown. Then substituting in (15) these values of $c, c', \frac{dy}{ds}, \frac{dx}{ds}$, it becomes

$$r=\frac{h}{2}\Big(\sin t\sin b+\cos t\cos b\Big)=\frac{h}{2}\cos(t-b), \tag{16}$$

which is the intrinsic equation of the cycloid, $h$ being equal to eight times the radius of the generating circle, and $b$ the angle made with the axis of $x$ by the chord joining the cusps.

**31.** Let us next examine the sign of the terms of the second order. Since those of the first order vanish, (4) becomes

$$\delta U = -\frac{1}{2}\int_{x_0}^{x_1}\left\{\frac{4(1+3y'^2)}{y''}\delta y'^2 - \frac{8y'(1+y'^2)}{y''^2}\delta y'\delta y'' + \frac{2(1+y'^2)^2}{y''^3}\delta y''^2\right\}dx$$

$$= \int_{x_0}^{x_1}\frac{-1}{2y''}\left\{4(1+y'^2+2y'^2)\delta y'^2 - \frac{8y'(1+y'^2)}{y''}\delta y'\delta y'' + \frac{2(1+y'^2)^2}{y''^2}\delta y''^2\right\}dx$$

$$= \int_{x_0}^{x_1}\frac{-1}{y''}\left\{2(1+y'^2)\delta y'^2 + 4y'^2\delta y'^2 - 4y'\frac{1+y'^2}{y''}\delta y'\delta y'' + \frac{(1+y'^2)^2}{y''^2}\delta y''^2\right\}dx$$

$$= \int_{x_0}^{x_1}\frac{-1}{y''}\left\{2(1+y'^2)\delta y'^2 + \left(2y'\delta y' - \frac{1+y'^2}{y''}\delta y''\right)^2\right\}dx. \quad (17)$$

But since the axis of $x$ is so taken as to render the cycloid concave to it, $y''$ is always negative, and therefore the factor $\frac{-1}{y''}$ is always positive, since the sign of each element depends upon that of this factor. We infer, therefore, that the cycloid is the curve required; although, because $y''$ becomes infinite at the two cusps, our investigation will perhaps be subject to some doubt if we are obliged to include either cusp within the range of integration.

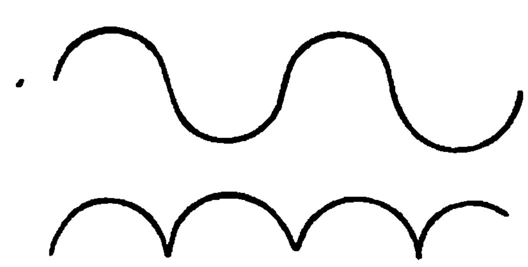
**32.** If we attempt to neglect the two conditions which are to hold at the limits, and to regard $\delta y'$ and $\delta y''$ as independent, we shall have the two equations

$$\frac{4y'(1+y'^2)}{y''} = 0, \qquad \frac{(1+y'^2)^2}{y''^2} = 0,$$

both of which give $y'' = \infty$, which cannot be true if the required curve is to be continuous. The seeming solution,

$y' = 0$, of the first equation must be rejected, because, if it could hold, the curve becoming a straight line would cause $y''$ to vanish also, and thus the equation would become indefinite.

**33.** It is evident that the cycloid will not give the least possible value of the area in question. For by joining arcs of cycloids, or even of circles, of indefinitely small radius, the area may be made as small as we please, as will appear by the subjoined figures:



We have, therefore, theoretically only a minimum in the technical sense hitherto explained.

In fact, the method here employed excludes all curves having either $y'$ or $y''$ infinite within the given range of integration; and it also enables us to compare the cycloid with such curves only as can be derived from it by any arbitrary indefinitely small changes in the values of $y'$ and $y''$. Still, under the conditions which we imposed upon the problem—viz., that the extreme points, and also the direction of the extreme tangents, should be fixed, and the subsequent condition that the required curve should be concave to the axis of $x$—there can, we think, be no doubt that the cycloid gives not only a minimum, but also the least value of the area in question.

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## SECTION II.

### *CASE IN WHICH THE LIMITING VALUES OF X ONLY ARE GIVEN.*

**34.** The reader having now become somewhat familiar with the general method of the calculus of variations, we shall next present some theoretical considerations, which are usually advanced before the discussion of problems is attempted.

*Suppose we wish to determine the conditions which will render $U$ a maximum or minimum, where*

$$U = \int_{x_0}^{x_1} f(x, y, dx, dy, d^2y, \text{etc.}).$$

Then it will be found, as in the preceding examples, that $U$ can be reduced to the form

$$U = \int_{x_0}^{x_1} V dx,$$

where $V$ is some function of $x, y, y', y''$, etc.

Now change $y$ into $y + \delta y$, $y'$ into $y' + \delta y'$, etc., $x$ remaining unaltered. Let $V$, in consequence of these changes, which are indefinitely small, become $V'$, and $U$ become $U'$. Then we shall have

$$U' = \int_{x_0}^{x_1} V' dx.$$

Also let $U' - U$ be denoted by $\delta U$, and $V' - V$ by $\delta V$. Then if $\delta y$, $\delta y'$, etc., be indefinitely small, $\delta U$ and $\delta V$ will also be indefinitely small. It is clear also that we shall have

$$U' - U \text{ or } \delta U = \int_{x_0}^{x_1} V' dx - \int_{x_0}^{x_1} V dx$$

$$= \int_{x_0}^{x_1} (V' - V)\, dx = \int_{x_0}^{x_1} \delta V dx. \tag{1}$$

Now if we develop $\delta V$ by Taylor's Theorem, it becomes

$$\delta V = \frac{dV}{dy}\delta y + \frac{dV}{dy'}\delta y' + \frac{dV}{dy''}\delta y'' + \text{etc.}$$

$$+ \frac{1}{2}\Big(A\,\delta y^2 + 2B\,\delta y\,\delta y' + C\,\delta y'^2 + 2D\,\delta y\,\delta y'' + \text{etc.}\Big), \tag{2}$$

in which $\frac{dV}{dy}$, $\frac{dV}{dy'}$, etc., are the partial differential coefficients of $V$ with respect to $y$, $y'$, etc.; and $A, B, C, D$, etc., are the second partial differential coefficients of $V$ with respect to the quantities whose variations immediately follow them. Substituting this value of $\delta V$ in (1), it becomes

$$\delta U = \int_{x_0}^{x_1} \left\{ \frac{dV}{dy}\,\delta y + \frac{dV}{dy'}\,\delta y' + \frac{dV}{dy''}\,\delta y'' + \text{etc.} \right\} dx$$

$$+ \frac{1}{2}\int_{x_0}^{x_1} (A\,\delta y^2 + 2\,B\,\delta y\,\delta y' + C\,\delta y'^2 + 2\,D\,\delta y\,\delta y'' + \text{etc.})\,dx. \quad (3)$$

Now, by our previous reasoning, the first integral must vanish for either a maximum or a minimum, while the second integral must become negative for a maximum and positive for a minimum.

**35.** It has probably been observed that our treatment of the terms of the first order has been quite uniform, while our treatment of those of the second order has differed in nearly every case. The general discussion of this latter part of the problem, or, as it is called, the *discrimination of maxima and minima*, is the most difficult of all the subjects connected with the calculus of variations. Although the foundations had been laid by Legendre and Lagrange, and the problem could be solved in certain cases, still no general method was known prior to the year 1837, when Jacobi published a theorem, which we shall explain hereafter, and which reduces this portion of our investigation also to a uniform rule.

We shall, therefore, at present speak of $\delta U$ as involving terms of the first order only, except when the contrary is expressly stated.

**36.** Let us now consider more generally than hitherto the equation $\delta U = 0$.

By (3) this becomes

$$\delta U = \int_{x_0}^{x_1} \left\{ \frac{dV}{dy}\delta y + \frac{dV}{dy'}\delta y' + \frac{dV}{dy''}\delta y'' + \text{etc.} \right\} dx = 0, \quad (4)$$

and this equation is true whether the values of $y, y', y''$, etc., at the limits are fixed or not, it being merely required that the limiting values of $x$ only should be fixed. Now by means of the known relations given in formulæ (A), (B) and (C) we can, by integration by parts, transform any term in (4) until it shall consist of terms free from the sign of integration, and an integral involving $\delta y\, dx$.

Let $N, P, Q, R, S$, etc., be the coefficients of $\delta y, \delta y', \delta y''$, etc., in (4), and consider for example the term

$$S\delta y^{\text{iv}} = \frac{Sd^4\delta y}{dx^4}.$$

We have

$$\int \frac{Sd^4\delta y}{dx^4} dx = \frac{Sd^3\delta y}{dx^3} - \int \frac{dS}{dx}\frac{d^3\delta y}{dx^3} dx,$$

$$-\int \frac{dS}{dx}\frac{d^3\delta y}{dx^3} dx = -\frac{dS}{dx}\frac{d^2\delta y}{dx^2} + \int \frac{d^2S}{dx^2}\frac{d^2\delta y}{dx^2} dx,$$

$$\int \frac{d^2S}{dx^2}\frac{d^2\delta y}{dx^2} dx = \frac{d^2S}{dx^2}\frac{d\delta y}{dx} - \int \frac{d^3S}{dx^3}\frac{d\delta y}{dx} dx,$$

$$-\int \frac{d^3S}{dx^3}\frac{d\delta y}{dx} dx = -\frac{d^3S}{dx^3}\delta y + \int \frac{d^4S}{dx^4}\delta y\, dx.$$

$$\int_{x_0}^{x_1} S\delta y^{\text{iv}} dx = \left(S\delta y''' - \frac{dS}{dx}\delta y'' + \frac{d^2S}{dx^2}\delta y' - \frac{d^3S}{dx^3}\delta y\right)_1$$
$$-\left(S\delta y''' - \frac{dS}{dx}\delta y'' + \frac{d^2S}{dx^2}\delta y' - \frac{d^3S}{dx^3}\delta y\right)_0$$
$$+\int_{x_0}^{x_1} \frac{d^4S}{dx^4}\delta y\, dx.$$

Integrating the other terms in $\delta U$ in a similar manner, collecting and arranging the results, we have

$$\delta U = \left(P - \frac{dQ}{dx} + \frac{d^2R}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right)_1 \delta y_1$$

$$- \left(P - \frac{dQ}{dx} + \frac{d^2R}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right)_0 \delta y_0$$

$$+ \left(Q - \frac{dR}{dx} + \frac{d^2S}{dx^2} - \text{etc.}\right)_1 \delta y_1'$$

$$- \left(Q - \frac{dR}{dx} + \frac{d^2S}{dx^2} - \text{etc.}\right)_0 \delta y_0'$$

$$+ \left(R - \frac{dS}{dx} + \text{etc.}\right)_1 \delta y_1'' - \left(R - \frac{dS}{dx} + \text{etc.}\right)_0 \delta y_0''$$

$$+ (S - \text{etc.})_1 \, \delta y_1''' - (S - \text{etc.})_0 \, \delta y_0''' + \text{etc.}$$

$$+ \int_{x_0}^{x_1} \left\{ N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} \right\} \delta y \, dx, \quad (5)$$

in which $\frac{dP}{dx}$, $\frac{d^2Q}{dx^2}$, etc., are the total differentials of these quantities with respect to $x$.

Finally, for convenience, (5) may be written thus:

$$\delta U = L - \int_{x_0}^{x_1} M \, \delta y \, dx, \quad (6)$$

and this equation holds, whether the values of $\delta y$, $\delta y'$, $\delta y''$, etc., at the limits vanish, as we have hitherto supposed, or not; the limiting values of $x$ only being required to remain fixed.

**37.** We see that $\delta U$ in (6) consists of two classes of terms which are essentially different; the first depending solely upon the values which the quantities $\delta y$, $\delta y'$, etc., and $P$, $Q$, $R$, etc., with their total differential coefficients, may have at the limits; while the second is an integral involving the general values of these quantities. Now since $\delta U$ must vanish when $U$ is to be a maximum or a minimum, let us consider these two parts of $\delta U$ separately in this case.

Write, for convenience,

$$L = h_1\,\delta y_1 - h_0\,\delta y_0 + i_1\,\delta y_1' - i_0\,\delta y_0' + j_1\,\delta y_1'' - j_0\,\delta y_0'' + \text{etc.} \qquad (7)$$

Then it is plain that the several quantities $\delta y_1$, $\delta y_0$, $\delta y_1'$, $\delta y_0'$, etc., are entirely in our power; that is, we may impose at the limits any conditions we please, so long as all the variations are indefinitely small and $x_0$ and $x_1$ remain immutable. It is likewise clear that the quantities $h_0$, $h_1$, $i_0$, $i_1$, etc., are not in our power. For suppose the equation

$$\delta U = L + \int_{x_0}^{x} M\,\delta y\,dx = 0 \qquad (8)$$

to have been solved so as to give $y$ as a function of $x$, say $f(x)$. Then this equation would be a solution of the problem to find the value of $y$, or the equation of a plane curve, which would render $U$ a maximum or a minimum; and as we wish to compare only this primitive with its derived curves, we must consider $h_0$, $h_1$, etc., as referring to this primitive and to the given limits only. These quantities can therefore, so soon as the equation of the curve and the values of $x_0$ and $x_1$ are known, be found.

Hence if $h_0$, $h_1$, $i_0$, $i_1$, etc., do not severally vanish, we can make $L$ assume any infinitesimal value we please by suitably choosing $\delta y_0$, $\delta y_1$, $\delta y_0'$, $\delta y_1'$, etc. But if the solution $y = f(x)$ cause these quantities to severally vanish, $L$ must become zero also, and no other condition will cause $L$ to vanish necessarily without restricting the values of $\delta y_0$, $\delta y_1$, $\delta y_0'$, etc.

**38.** Let us now consider the second term,

$$\int_{x_0}^{x_1} M\,\delta y\,dx.$$

In this integral $\delta y$ is wholly in our power, being subject only to the condition that neither it nor any of its differential coefficients, to the *nth* inclusive, shall become appreciable within the range of integration, $y^{(n)}$ being the highest differential coefficient in $V$. In other words, $\delta y$ may be any arbitrary function of $x$ which fulfils these conditions, or it need not even be the same function throughout the entire range of integration.

On the other hand, $M$ is not in our power, but will, as in the case of $h_0$, $h_1$, etc., depend upon the equation $y = f(x)$. Hence if $M$ be not necessarily zero throughout the given limits of integration, the integral will be wholly in our power, and we may, by suitably varying $y$, make it assume any infinitesimal value we please. But if the solution $y = f(x)$ reduce $M$ to zero throughout $U$, then the integral itself, being definite, must become zero; and it will not necessarily vanish under any other condition, so long as $\delta y$ is wholly unrestricted.

**39.** Resuming equation (8), we have

$$L = -\int_{x_0}^{x_1} M\,\delta y\,dx. \qquad (9)$$

Now if the solution $y = f(x)$ be such as to cause the quantities $h_0$, $h_1$, $i_0$, $i_1$, etc., and also $M$ to severally vanish, then each member of (9) will likewise vanish, and no difficulty will occur. But if the proposed solution be not able to fulfil all these conditions, (9) becomes an impossible equation. For inasmuch as $L$ and $\int_{x_0}^{x_1} M\,\delta y\,dx$ are no longer necessarily zero, it would in effect imply, as Prof. Jellett has remarked, "that the integral of an arbitrary function may be expressed (without deter-

mining or even restricting its general form) in terms of the limiting values of itself and a certain number of its differential coefficients. This is manifestly untrue."

We conclude, then, that it is necessary to the existence of a maximum or minimum not only that $L$ and $M$ shall vanish, but that each of the quantities $h_0$, $h_1$, $i_0$, $i_1$, etc., and $M$, shall become zero.

**40.** Although the truth of the preceding principles would appear to be sufficiently evident, yet Strauch, one of the most elaborate writers on the calculus of variations, asserts that it cannot be proved that $L$ and $\int_{x_0}^{x_1} M\,\delta y\,dx$ must severally vanish; and as this is a point of the highest importance, and of some difficulty, we have given it more attention than it has generally received hitherto. Strauch is, however, compelled to admit that we do obtain correct results by this method; and there can, as Prof. Todhunter states, be no doubt that the principle is sound.

**41.** Before proceeding further we will apply the foregoing theory to the solution of some examples.

### Problem IV.

*Let $V$ be any function of $y''$ and constants only, and let it be required to determine the relations which must subsist between $x$ and $y$ in order to maximize or minimize the expression*

$$U = \int_{x_0}^{x_1} V dx,$$

*$x_0$ and $x_1$ only being fixed.*

We have

$$\delta U = \int_{x_0}^{x_1} \frac{dV}{dy''}\,\delta y''\,dx = \int_{x_0}^{x_1} Q\,\delta y''\,dx = 0.$$

Then transforming $\delta U$, as just explained, and denoting by accents total differentials, we have

$$\delta U = -Q_1'\delta y_1 + Q_0'\delta y_0 + Q_1\delta y_1' - Q_0\delta y_0'$$

$$+\int_{x_0}^{x_1} Q''\,\delta y\,dx = 0. \qquad (1)$$

Whence, since $M$ must vanish, we have

$$Q'' = 0, \qquad Q' = c, \qquad Q = cx + c'. \qquad (2)$$

If we had supposed the values $y$ and $y'$ at the limits to be given as in former examples, the solution could be carried no further without determining the form of $V$. But since $\delta y_0$, $\delta y_1$, $\delta y_0'$, $\delta y_1'$ are not necessarily zero, we must, from the preceding discussion, have the coefficients of these quantities severally zero. Hence $Q_1' = 0$, $Q_0' = 0$, $Q_1 = 0$, $Q_0 = 0$. From the third and fourth of these equations, combined with (2), we have $cx_1 + c' = 0$, $cx_0 + c' = 0$, $c(x_1 - x_0) = 0$. Whence $c = 0$, and then $c' = 0$. Therefore the last of equations (2) gives $Q = 0$.

If this equation is to hold throughout $U$, $y''$ must be constant, although it may have several constant values. Let $a$ be one of the roots of the equation $Q = 0$. Then, as $y'' = a$, by integration we obtain

$$y = \frac{ax^2}{2} + bx + b', \qquad (3)$$

the equation of a parabola; or of a straight line if $a$ should happen to become zero.

The constants $b$ and $b'$ cannot be determined so long as the values of $y_0$ and $y_1$ are not fixed. For it is easy to see that the equations $Q_1' = 0$ and $Q_0' = 0$ furnish no new equations

of condition, because they follow from $Q = 0$, and any values of $b$ and $b'$ which satisfy the latter will also satisfy the former two.

Owing to its simplicity, we may also examine the term of the second order, which is

$$\frac{1}{2}\int_{x_0}^{x_1}\frac{d^2Q}{dy''^2}\,\delta y''^2 dx.$$

Since $y''$ is a constant, $\frac{d^2Q}{dy''^2}$, which is some function of $y''$, must be also a constant, say $A$; then, since the terms of the first order vanish, we may write

$$\delta U = \frac{A}{2}\int_{x_0}^{x_1}\delta y''^2 dx,$$

which shows that we have a maximum or minimum according as $A$ is negative or positive.

### Problem V.

**42.** *It is required to maximize or minimize the expression*

$$U = \int_{x_0}^{x_1}(y''^2 - 2y)\,dx = \int_{x_0}^{x_1} V dx,$$

*the limiting values of $x$ only being given.*

We have

$$\delta U = \int_{x_0}^{x_1}(2y''\delta y'' - 2\delta y)\,dx = -\,2y_1'''\,\delta y_1 + 2y_0'''\,\delta y_0$$

$$+\,2y_1''\,\delta y_1' - 2y_0''\,\delta y_0' + \int_{x_0}^{x_1}(2y^{\text{iv}} - 2)\,\delta y\,dx = 0. \qquad (1)$$

Whence equating $M$ to zero, and integrating, we have

$$y^{\text{iv}} = 1, \tag{2}$$

$$y''' = x + a, \tag{3}$$

$$y'' = \frac{x^2}{2} + ax + b, \tag{4}$$

$$y' = \frac{x^3}{6} + \frac{ax^2}{2} + bx + c, \tag{5}$$

$$y = \frac{x^4}{24} + \frac{ax^3}{6} + \frac{bx^2}{2} + cx + d. \tag{6}$$

Now if $\delta y_0$ and $\delta y_1$ be unrestricted, we must have, from (1), $y_1''' = 0, y_0''' = 0$, which give, in (3), $x_0 + a = 0$, and $x_1 + a = 0$, which are impossible equations, since $x_0$ and $x_1$ are not to be equal. Whence we conclude that the solution will not be possible unless we restrict $\delta y_0$ and $\delta y_1$, so that $y_0'''$ and $y_1'''$ need not severally vanish.

We will now suppose $y_0$ and $y_1$ to be given, but $y_0'$ and $y_1'$ to be unrestricted. Then, from (1), we must have $y_0'' = 0$, $y_1'' = 0$, which would give, in (4),

$$\frac{x_1^2}{2} + ax_1 + b = 0, \tag{7}$$

$$\frac{x_0^2}{2} + ax_0 + b = 0. \tag{8}$$

From these equations we readily obtain

$$a = -\frac{1}{2}(x_0 + x_1), \tag{9}$$

$$b = \frac{x_0\, x_1}{2}. \tag{10}$$

Now suppose, for simplicity, that we take $x_1$ equal to any constant $e$, and $x_0$ to $-e$. Then (9) and (10) give $a = 0$, $b = -\frac{e^2}{2}$, and (6) would become

$$y = \frac{x^4}{24} - \frac{e^2x^2}{4} + cx + d. \tag{11}$$

But it will be remembered that this equation is only admissible on the supposition that we are able to make $h_1\delta y_1 - h_0\delta y_0$ vanish; and as $h_1$ and $h_0$ cannot severally vanish, this is accomplished by fixing the values of $y_1$ and $y_0$, and the assignment of these values will afford us the conditions for determining the remaining constants. Equation (11) now gives

$$y_1 = \frac{-5e^4}{24} + ce + d, \tag{12}$$

$$y_0 = \frac{-5e^4}{24} - ce + d. \tag{13}$$

Whence we obtain

$$c = \frac{y_1 - y_0}{2e}, \tag{14}$$

$$d = y_1 + \frac{5e^4}{24} - \frac{y_1 - y_0}{2}. \tag{15}$$

Suppose, for still greater simplicity, we take the fixed points on the axis of $x$. Then (14) and (15) give $c = 0$, $d = \frac{5e^4}{24}$, and we shall have, finally,

$$y = \frac{x^4}{24} - \frac{e^2x^2}{4} + \frac{5e^4}{24}.$$

But suppose, as usual, the limiting values of $y$ and $y'$ were both given, and let us consider the particular case in which

we have $x_1 = e$, $x_0 = -e$, $y_1 = 0$, $y_0 = 0$, $y_1' = 1$, $y_0' = -1$. Then, from (5), we have

$$y_1' = \frac{x_1^3}{6} + \frac{ax_1^2}{2} + bx_1 + c, \tag{17}$$

$$y_0' = \frac{x_0^3}{6} + \frac{ax_0^2}{2} + bx_0 + c. \tag{18}$$

Then eliminating $c$ between (17) and (18), we have

$$b = \frac{6 - e^3}{6e}, \tag{19}$$

and from the same equations we obtain

$$a = \frac{-2c}{e^2}. \tag{20}$$

Moreover, substituting in turn $e$ and $-e$ for $x$ in (6), we have

$$y_1 = \frac{e^4}{24} + \frac{ae^3}{6} + \frac{be^2}{2} + ce + d, \tag{21}$$

$$y_0 = \frac{e^4}{24} - \frac{ae^3}{6} + \frac{be^2}{2} - ce + d. \tag{22}$$

Eliminating $d$, we have

$$\frac{ae^2}{3} + 2ce = 0. \tag{23}$$

Substituting for $a$ its value from (20), we find $c = 0$, whence also $a = 0$; and again substituting these values with that of $b$, (21) gives

$$d = \frac{e^4}{24} - \frac{e}{2}.$$

Now substituting these values in (6), we have, finally,

$$y = \frac{x^4}{24} + \frac{6 - e^3}{12e}x^3 + \frac{e^4}{24} - \frac{e}{2}.$$

The term of the second order is merely

$$\int_{x_0}^{x_1} \delta y''^2 dx,$$

which is of course positive, thus giving a minimum. That is, any solution which reduces the terms of the first order to zero will render $U$ a minimum.*

**43.** Now resume for a moment the consideration of Prob. I. There we have

$$h_1 = \left\{ \frac{y'}{\sqrt{1 + y'^2}} \right\}_1, \qquad h_0 = \left\{ \frac{y'}{\sqrt{1 + y'^2}} \right\}_0,$$

which give $y_0' = 0$, and $y_1' = 0$. But since we know from the general solution that $y' = a$, these two conditions are in reality only one, $a = 0$. Hence if no restrictions be imposed except that $x_0$ and $x_1$ shall be fixed, the line must be parallel to the axis of $x$.

But the constant $b$ cannot be determined in this case. Indeed it is evident that the straight line parallel to $x$ is shorter than any other curve, or straight line even, which can be drawn having $x_0$ and $x_1$ as the abscissæ of its extremities, and that hence our first result is confirmed. Moreover, since the length of this line will be the same, whatever be its distance from the axis of $x$, the value of $b$ can have no effect upon its length, and therefore ought to remain undetermined. If, however, the co-ordinates of one of its extremities be given, the line becomes a parallel to $x$ through that fixed point, and $b$ is determined.

* The last two examples are from the Adams Essay, by Prof. Todhunter (p. 15), but have been considerably elaborated.

**44.** Next consider Prob. II., Case 1. There we find

$$h_1 = \left\{ \frac{y'}{\sqrt{(1+y'^2)x}} \right\}_1, \qquad h_0 = \left\{ \frac{y'}{\sqrt{(1+y'^2)x}} \right\}_0.$$

But from equation (8), Art. 17, if we make $h_1$ or $h_0$ zero, we see that the equation

$$\frac{y'}{\sqrt{(1+y'^2)x}} = 0$$

must hold throughout the curve, and this gives $y' = 0$, which, as it denotes the vertical, is the true solution. For if a particle be merely required to descend from one horizontal plane to another, it will do so along the vertical sooner than along any other line. The equation of this vertical is $y = b$, in which the value of $b$ can have no effect upon the time of descent, and therefore remains undetermined, as it should.

Next consider the second case of the same problem. There we have

$$h_1 = \left\{ \frac{y'}{\sqrt{(1+y'^2)y}} \right\}_1, \qquad h_0 = \left\{ \frac{y'}{\sqrt{(1+y'^2)y}} \right\}_0.$$

The first of these equations gives $y_1' = 0$; and since $\sqrt{(1+y'^2)y} = \sqrt{2r}$, $r$ being the radius of the generating circle, we have

$$\left\{ \frac{y'}{\sqrt{(1+y'^2)y}} \right\}_0 = \frac{y_0'}{\sqrt{2r}},$$

and this, if equated to zero, will give $y_0' = 0$, which is evidently impossible. Hence $h_0$ cannot be zero; and to make the term $h_0 \delta y_0$ vanish, we must assign the value of $y_0$.

Now it will be remembered that the general solution was a cycloid, having a cusp at the starting-point of the particle, and that $b$ was merely the value of $y_0$, which is now determined. Moreover, since we have just found that the tangent to this cycloid at the point which is not fixed must be par-

allel to the axis of $x$, it follows that its vertex must be at this point. Hence the generating circle must be such that it would roll through a semicircle while its centre was describing the distance $x_1 - x_0$, and therefore we have

$$r = \frac{x_1 - x_0}{\pi}.$$

**45.** Let us in the last place consider Prob. III. If we could have fully integrated equation (10), Art. 30, the integral would have involved four constants, and for determining these constants we would have $y_1'$, $y_0'$, $y_1$, $y_0$ equal to four assigned quantities. It would, however, be too tedious to discuss this case in detail, and we will next suppose the values of $y_0$ and $y_1$ to be fixed, while those of $y_0'$ and $y_1'$ are variable. Then equating to zero the coefficients of $\delta y_1'$ and $\delta y_0'$, we shall have

$$\left\{\frac{(1+y'^2)^2}{y''^2}\right\}_1 = 0, \qquad \left\{\frac{(1+y'^2)^2}{y''^2}\right\}_0 = 0;$$

and since $\sqrt{1+y'^2}$ cannot be zero, $y_1''$ and $y_0''$ must each be infinite, thus giving the cycloid cusps at the two fixed points. Let $b$ denote the angle which the line joining these cusps makes with the axis of $x$. Then $b$ is identical with $b$ of equation (16), Art. 30, and is at once determined, its tangent being $\dfrac{y_1 - y_0}{x_1 - x_0}$. Then, also,

$$\frac{h}{8} = r = \frac{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}}{2\pi}.$$

Let us now suppose the values of $y_1$ and $y_0$ to be unrestricted. Then we must equate the coefficients of $\delta y_1$ and $\delta y_0$ severally to zero, which will give the equation

$$\left\{\frac{4y'(1+y'^2)}{y''} + \frac{d}{dx}\frac{(1+y'^2)^2}{y''^2}\right\}_1 = 0,$$

and a similar equation for the lower limit. But from equation (11), Art. 30, the first member of the last equation equals $-c$, making $c$ in this case zero. Therefore equation (13) of the same article becomes

$$\frac{(1+y'^2)^2}{y''} = -\frac{d}{2}.$$

Now $d$ cannot vanish. For if it can, we must either have $\sqrt{1+y'^2}=0$, which would render $y'$ imaginary, or $y''$ must be infinite throughout the curve, which is also inadmissible. But if $\delta y_1'$ and $\delta y_0'$ do not vanish, we must, as we have just seen, have $y_1''$ and $y_0''$ infinite. It follows, therefore, that $y_1'$ and $y_0'$ must become infinite, as $d$ would otherwise vanish.

We conclude, then, that the cycloid must in this case be so placed as to have the line joining its cusps parallel to the axis of $x$. Then we shall evidently have

$$r = \frac{x_1 - x_0}{2\pi},$$

while the constant angle $b$ of equation (16), Art. 30, will become zero, and it is easy to show also that $d = 8r = h$.

**46.** It is evident that none of the results of the preceding articles could be confirmed as maxima or minima without an examination of the sign of the terms of the second order, because even if those terms were shown to be certainly positive or negative, in any particular problem, by making any of the variations $\delta y_0$, $\delta y_1$, $\delta y_0'$, $\delta y_1'$, etc., zero, it would not follow that we could be certain of the same sign when those restrictions were removed or modified.

But it will be remembered that in the problems thus far discussed we have, with the exception of Case 2, Prob. II., been able to determine the sign of the terms of the second order without imposing any restriction upon the variations of $y$ and $y'$ at the limits. The only result, then, which we

have to confirm is this: that when the starting-point of the particle is given, its terminal point being restricted to have a given abscissa $x_1$, the curve of quickest passage from $x_0$ to $x_1$ will be a cycloid with a cusp at the first point, and its vertex at the second. An examination of equations (12) and (13), Art. 27, will show that if we had not supposed $\delta y_1$ and $\delta y_0$ to be zero, equation (19) of the same article would have become

$$\delta U = \frac{1}{2\sqrt{a}}\left\{\left(\frac{-y'}{2y}\,\delta y^2\right)_1 + \left(\frac{y'}{2y}\,\delta y^2\right)_0\right.$$

$$\left. + \int_{x_0}^{x_1}\left(\frac{\delta y^2}{2y^2} + \frac{y}{a}\,\delta y'^2\right)dx\right\} = 0,$$

in which the integral is positive as before. But by hypothesis $\delta y_0 = 0$, and as $y'$ vanishes at the vertex, while $y$ becomes $a$ or $2r$, we have $\left(\frac{y'}{2y}\right)_1 = 0$. Hence both terms without the sign of integration vanish, and we have a minimum as before.

**47.** We may now proceed without difficulty to that general discussion of the terms of the first order which is usually, but unadvisedly we think, presented prior to the discussion of particular problems.

Assume the equation $U = \int_{x_0}^{x_1} V dx$, where $V$ is any function of $x, y, y' \ldots . y^{(n)}$, and let it be required to determine what function $y$ must be of $x$ in order to render $U$ a maximum or minimum. Then finding $\delta U$, and transforming it by integration as far as possible, and then equating to zero severally the coefficients of $\delta y_1$, $\delta y_0$, etc., together with $M$, which is the coefficient of $\delta y\, dx$ under the integral sign, we obtain the equations $h_1 = 0$, $h_0 = 0$, $i_1 = 0$, $i_0 = 0$, etc., and also $M = 0$, where, as will be remembered,

$$M = N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \text{etc.},$$

all the differentials being total, and $N$, $P$, $Q$, etc., being the partial differential coefficients of $V$ with respect to $y$, $y'$, $y''$, etc.

Now the equation $M = 0$ will, in general, be a differential equation of the order $2n$, because its last term will be

$$\pm \frac{d^n}{dx^n} \frac{dV}{dy^{(n)}},$$

which will usually involve

$$\frac{d^n y^{(n)}}{dx^n} = (y^{(n)})^{(n)} = y^{(2n)}.$$

Hence the complete integral of this equation must usually contain $2n$ arbitrary constants, and may be supposed to be put under the form

$$y = f(x, c_1, c_2, \dots . c_{2n}) = f. \tag{1}$$

Now since every solution of our problem must satisfy the equation $M = 0$, it must also be comprised in (1), which establishes a general relation between $x$ and $y$, or, in other words, gives us some plane curve; which relation or curve is, however, capable of great modification, by adjusting suitably the values of these $2n$ arbitrary constants.

**48.** If now we examine the equations $h_1 = 0$, $h_0 = 0$, etc., which we may call the equations at the limits, we shall find that their number is also $2n$. Moreover, these equations, not holding throughout the curve, do not establish any general relation between $x$ and $y$, as did the equation $M = 0$, but merely fix the conditions which the required curve must fulfil at the limits. This is as it should be. For if the equations $h = 0$, $i = 0$, etc., could be supposed to hold throughout the curve, they would each establish a relation between $x$ and $y$, and unless these relations should happen to agree with each

other, and also with that derived from the equation $M = 0$, which would seldom if ever occur, the solution would become nugatory.

Now suppose the complete integral of the equation $M = 0$ were obtained, and expressed as in (1). Then if the form of $f$ were known, we could form the expressions $h_1$, $h_0$, $i_1$, etc., and these expressions would all be known functions of either $x_1$ or $x_0$, together with some of the $2n$ arbitrary constants, no variable entering these functions, because $x_0$ and $x_1$, being assigned quantities, may be regarded as constants also.

We see then that in the equations $h_1 = 0$, $h_0 = 0$, etc., we have $2n$ equations between $x_0$ and $x_1$ which are assigned, and $2n$ arbitrary constants, and should therefore be able to determine these $2n$ constants in terms of the known constants $x_0$ and $x_1$.

Now suppose the limiting values of $y_1$ and $y_0$ were given. Then, since the variations of these quantities would become zero, $h_1$ and $h_0$ would no longer necessarily vanish. But in this case it is evident that the two equations thus lost would be replaced by the equations $y_1 = f(x_1, c_1, c_2 \ldots . c_{2n}) = f_1$, and $y_0 = f(x_0, c_1, c_2 \ldots . c_{2n}) = f_0$; and as $y_1$ and $y_0$ are now supposed to have assigned values, the number of the equations for the determination of the $2n$ constants remains, as before, $2n$. In like manner, if $\delta y_1'$ and $\delta y_0'$ should become zero, the conditions $i_1 = 0$ and $i_0 = 0$ would disappear. But to supply their place we would have the equation

$$y_1' = \frac{d}{dx} f(x_1, c_1, c_2, \ldots . c_{2n}) = f',$$

and a similar equation for the lower limit, $y_1'$ and $y_0'$ being now assigned constants also; so that we still have, as before, $2n$ ancillary equations.

Suppose, lastly, that any of the variations $\delta y_1$ $\delta y_0$, $\delta y_1'$, etc., were connected by given equations, and suppose there were $m$ such equations. Then if we should express as many of the

variations as possible in terms of the remaining variations, and then equate to zero the coefficients of the several variations in the reduced system, it is plain that our ancillary equations would be only $2n - m$ in number. But since we have the $m$ equations between certain variations, we are evidently able to form new systems of independent variations in such a manner as to obtain $m$ more equations between $x_1$, $x_0$, and the $2n$ constants.

Thus we see that, theoretically at least, the terms at the limits furnish us with $2n$ equations for the determination of the $2n$ arbitrary constants, which would in general occur in the complete integral of the equation $M = 0$, and that whatever condition reduces the number of the original equations, by annulling or combining two or more of them, will at the same time furnish in their place as many new equations for the determination of these constants as have been removed.

**49.** The preceding considerations, which are theoretical, require some modification, first as regards the terms at the limits, and second as regards the equation $M = 0$. With regard to the terms at the limits, it has probably been noticed that it has not been in general possible to satisfy all the equations $h_1 = 0$, $h_0 = 0$, etc., as some of these equations become conflicting. But even in these cases we can, as we have seen, generally obtain $2n$ harmonious equations by restricting one or more of the variations; as, for example, by supposing $\delta y_1$, $\delta y_0$, or $\delta y_1'$, etc., to vanish.

In fact, the occurrence of these conflicting equations denotes merely that the problem in its present form is not capable of solution, and as it might be foreseen that such questions would present themselves, the occurrence of these conflicting equations would naturally be expected.

**50.** The following exceptions may be regarded as due to the nature of the equation $M = 0$, although they properly arise from the nature of the function $V$.

Exception 1. Suppose $N$ to vanish in the equation $M = 0$, which would of course happen if $y$ did not explicitly enter $V$. Then we would have

$$M = -\frac{dP}{dx} + \frac{d^2Q}{dx^2} - \text{etc.} = 0,$$

whence

$$P - \frac{dQ}{dx} + \frac{d^2R}{dx^2} - \text{etc.} = c.$$

But the first member of the last equation equals $h$; and as $h$ must vanish at either limit unless the values of $y_1$ and $y_0$ be assigned, we have $c = 0$; and since the equations $h_1 = 0$ and $h_0 = 0$ are each satisfied by this value of $c$, they furnish no new condition for the determination of any other constant which may enter the complete integral of the equation $M = 0$. Thus the conditions furnished by the terms at the limits are in this case reduced to $2n - 1$, two of them having become identical. If, however, the value of either $y_1$ or $y_0$ be assigned, this will furnish a new equation of condition which will compensate for that which was lost.

This case is fully exemplified by the discussion of Prob. I. in Art. 43 and Prob. II., Case 1, in Art. 44.

Similarly, suppose $V$ to contain neither $y$ nor $y'$. Then we would have

$$M = \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \text{etc.} = 0, \tag{1}$$

$$\frac{dQ}{dx} - \frac{d^2R}{dx^2} + \text{etc.} = a, \tag{2}$$

$$Q - \frac{dR}{dx} + \text{etc.} = ax + b. \tag{3}$$

Now if the limiting values of $y$ are variable, we have $h_1 = 0$ and $h_0 = 0$; and it is easy to see that in this case, as $P$ is want-

ing, the first member of (3) is $i$, and that of (2) is $-h$, and therefore we have $ax_1 + b = 0$, and $ax_0 + b = 0$, whence we find $a = 0$ and $b = 0$, and (3) becomes

$$Q - \frac{dR}{dx} + \frac{d^2S}{dx^2} - \text{etc.} = 0. \tag{4}$$

Now it must be remembered that this equation has been deduced solely from the conditions $i_1 = 0$ and $i_0 = 0$. But differentiating (4), we have

$$\frac{dQ}{dx} - \frac{d^2R}{dx^2} + \text{etc.} = 0, \qquad \text{or} \quad -h = 0.$$

Whence it appears that since the equations $h_1 = 0$, $h_0 = 0$, can, without involving any other relations, be deduced from the equations $i_1 = 0$, $i_0 = 0$, they furnish no new data for the determination of the constants which will be found in the complete integral of the equation $M = 0$. Hence in this case our ancillary equations will furnish but $2n - 2$ distinct conditions, thus leaving generally two constants undetermined, unless one or more additional equations be supplied by assigning the values of one or more of the quantities $y_1$, $y_0$, $y_1'$, $y_0'$, etc. This case is fully exemplified in the discussion of Prob. IV.

Generally, if the first $m$ of the quantities $y, y', y''$, etc., be wanting in $V$, while at the same time the variations of these quantities at the limits remain unrestricted, $m$ arbitrary constants in the general solution must also remain undetermined.

**51.** Exception 2. Suppose $V$ to contain only the first power of $y^{(n)}$, the highest differential coefficient which is involved. Then in this case the equation $M = 0$ cannot be of an order higher than $2n - 1$. For the last term in $M$ must be $\pm \frac{d^n}{dx^n} \frac{dV}{dy^{(n)}}$, and as only the first power of $y^{(n)}$ occurs in $V$, the

partial differential coefficient of $V$ with respect to $y^{(n)}$ will not contain that quantity at all. Whence it is evident that $M$ cannot be of an order $2n$; and indeed Prof. Jellett has shown that it cannot in this case rise above the order $2n - 2$ (see his page 46), but it does not seem necessary to reproduce his proof here.

Now in this case the equations at the limits will be, as before, $2n$ in number, while the constants in the complete integral of the equation $M = 0$ will not exceed in number $2n - 1$, and in *fact* will not exceed $2n - 2$. This seeming exception is, however, explained by the fact that in all such cases the integral $U$, or $\int_{x_0}^{x_1} Vdx$, is capable of being reduced by integration to the form $U = f_1 - f_0 + \int_{x_0}^{x_1} V'dx$, where $f_1$ and $f_0$ are quantities free from the sign of integration, while $V'$ does not contain any differential coefficient of $y$ higher than $y^{(n-1)}$; and we will next show that this reduction can be effected.

**52.** Let $y^{(n)}$ be the highest differential coefficient in $V$. Then, since its first power only occurs in $V$, we may write

$$V = wy^{(n)} + z, \tag{1}$$

where $w$ is that part of $V$ which is a factor of $y^{(n)}$, and $z$ the other terms of $V$, both being of course of a lower order than $y^{(n)}$. Then the equation $U = \int_{x_0}^{x_1} Vdx$ becomes

$$U = \int_{x_0}^{x_1} wy^{(n)}dx + \int_{x_0}^{x_1} zdx. \tag{2}$$

But we are evidently able to form the following equation:

$$\int wy^{(n)}dx = W + \int Zdx, \tag{3}$$

where $W$ and $Z$ are functions at present unknown. For this equation can, if in no other manner, always be formed thus:

$$\int wy^{(n)}dx = wy^{(n)}x + \int - \frac{d}{dx} wy^{(n)}.xdx. \qquad (4)$$

But $W$ and $Z$ can be so taken that the second member of (3) will contain no higher differential coefficient than $y^{(n-1)}$, because (3) can, in the following manner, be satisfied upon this assumption. First differentiate (3), and we shall have

$$wy^{(n)} = Z + \frac{dW}{dx} + \frac{dW}{dy}y' + \frac{dW}{dy'}y'' + \text{etc.} + \frac{dW}{dy^{(n-1)}}y^{(n)}, \qquad (5)$$

which must be the complete differential of (3) if our assumption be true, but not otherwise. But (5), and consequently (3), will be satisfied if we put

$$\frac{dW}{dy^{(n-1)}} = w, \qquad (6)$$

$$-Z = \frac{dW}{dx} + \frac{dW}{dy}y' + \text{etc.} + \frac{dW}{dy^{(n-2)}}y^{(n-1)}. \qquad (7)$$

Therefore $W$ is found by integrating $w$ with respect to $y^{(n-1)}$ only. Hence, finally, we have*

$$U = \int_{x_0}^{x_1} Vdx = W_1 - W_0 + \int_{x_0}^{x_1} (z + Z)dx$$

$$= W_1 - W_0 + \int_{x_0}^{x_1} V'dx. \qquad (8)$$

This case, then, is in reality no exception at all, because the

* This theorem is due to the great Euler (see Meth. Inven., pp. 62, 75), and has been nearly reproduced by Prof. Jellett on his page 46.

difficulty arises merely from the fact that the original integral had not been reduced to its lowest terms. For although we have not yet considered the class of problems to which this reduced form of $U$ belongs, it is easy to see that the equation $M=0$, resulting from $V'$ only, will not now be of an order exceeding $2n-2$, which is the result obtained by Prof. Jellett.

**53.** Exception 3. Let $V$ be of the form $yf+F$, where $f$ contains only quantities incapable of variation, e.g. $x$ and constants, and $F$ may contain any quantities except $y$. Then $N$ becomes simply $f$, and the equation $M=0$ will give the equations

$$\frac{dP}{dx}-\frac{d^2Q}{dx^2}+\text{etc.}=f,$$

$$P-\frac{dQ}{dx}+\frac{d^2R}{dx^2}-\text{etc.}=\int f dx=f'(x)+c. \qquad (1)$$

Now the first member of (1) equals $h$; and if we suppose $y_1$ and $y_0$ to be unrestricted, we must have $h_1=0$, $h_0=0$; and using these restrictions, (1) will give

$$\Big\{f'(x)\Big\}_1+c=0, \qquad (2)$$

and

$$\Big\{f'(x)\Big\}_0+c=0. \qquad (3)$$

But as the first members of (2) and (3) contain only one indeterminate constant, $c$, it will in general be impossible to satisfy both equations, and the problem in this form does not usually admit of a solution. But if we make $f$ zero, so that $V$ is any function not containing $y$, the problem becomes a case of Exception 1, and may or may not, according to its nature, be capable of a general solution, one constant at least

remaining undetermined. This exception is exemplified by Prob. V., in which $f = -2$, $F = y''^2$.

**54.** It is now evident that if we require that $U$ shall be a maximum or minimum, the calculus of variations will terminate its aid in the discussion by leaving us with a series of differential equations, that of the highest order holding true for all values of $x$ from $x_0$ to $x_1$, the others merely holding at the limits of integration. From the former of these equations, as it is general, the general solution must be obtained, and then the remaining or ancillary equations, not being general, must be satisfied, if they can be satisfied at all, by the assignment of suitable values to the constants which will occur in the general solution; or we may say that these ancillary equations determine the values of the constants.

The determination of these constants is not in general difficult when the complete integral of the equation is known; but this integral is often obtained with difficulty, and is sometimes altogether unobtainable. In fact, this difficulty is analogous to that which is frequently experienced in solving the final equation or equations of condition given by the differential calculus in the discussion of an ordinary problem of maxima or minima, except that in the former case the final equations are differential and must be solved by the calculus, while in the latter they are algebraic and must be solved by the *theory of equations*.

**55.** We shall next proceed to establish some principles regarding the integrability of the equation $M = 0$, and to deduce some formulæ which will be found useful in our subsequent discussions.

Suppose, in the first place, that the first $m$ of the quantities $N$, $P$, $Q$, $R$, etc., were wanting in the equation $M = 0$, which would of course happen if the first $m$ of the quantities $y$, $y'$, $y''$, $y'''$, etc., were wanting in $V$; then the equation $M = 0$ can be integrated at least $m$ times.

For let $m$ be 4, for example. Then we would have

$$M = \frac{d^4S}{dx^4} - \frac{d^5T}{dx^5} + \text{etc.} = 0,$$

which, being integrated four times, becomes

$$S - \frac{dT}{dx} + \text{etc.} = \frac{ax^3}{6} + \frac{bx^2}{2} + cx + d,$$

and similarly if $m$ were any other number.

**56.** Suppose, in the second place, that the independent variable $x$ does not occur explicitly in $V$; then the equation $M = 0$ can be integrated at least once. For since $V$ does not contain $x$, we have

$$\begin{aligned} dV &= Ndy + Pdy' + Qdy'' + Rdy''' + \text{etc.} \\ &= (Ny' + Py'' + Qy''' + Ry^{iv} + \text{etc.})\, dx. \qquad (1) \end{aligned}$$

Now substituting in the last member of (1) the value of $N$ derived from the equation $M = 0$, viz.,

$$N = \frac{dP}{dx} - \frac{d^2Q}{dx^2} + \text{etc.},$$

we shall have

$$\begin{aligned} dV = \left\{ Py'' + y' \frac{dP}{dx} \right\} dx + \left\{ Qy''' - y' \frac{d^2Q}{dx^2} \right\} dx \\ + \left\{ Ry^{iv} + y' \frac{d^3R}{dx^3} \right\} dx + \text{etc.} \qquad (2) \end{aligned}$$

But every parenthesis in (2) can be integrated by parts. Taking, for example, the third, and recollecting that

$$y^{iv}\, dx = dy''', \qquad y''dx = dy', \qquad y'dx = dy,$$

we have

$$\int Ry^{\text{iv}}\,dx = Ry''' - \int \frac{dR}{dx} y''' dx,$$

$$-\int \frac{dR}{dx} y''' dx = -\frac{dR}{dx} y'' + \int \frac{d^2R}{dx^2} y'' dx,$$

$$\int \frac{d^2R}{dx^2} y'' dx = \frac{d^2R}{dx^2} y' - \int \frac{d^3R}{dx^3} y' dx.$$

Hence

$$\int \left\{ Ry^{\text{iv}} + y' \frac{d^3R}{dx^3} \right\} dx = Ry''' - \frac{dR}{dx} y'' + \frac{d^2R}{dx^2} y'. \tag{3}$$

Integrating the remaining terms in a similar manner, we would have

$$V = c + Py' + \left\{ Qy'' - \frac{dQ}{dx} y' \right\}$$

$$+ \left\{ Ry''' - \frac{dR}{dx} y'' + \frac{d^2R}{dx^2} y' \right\} + \text{etc.}, \tag{A}$$

which equation is certainly of an order lower than that of the differential equation $M = 0$.

The following particular cases of this formula are given for convenience of reference:

*First.* If $V$ be a function of $y'$ only, we shall have, from (A),

$$V = c + Py'. \tag{B}$$

But since in this case $V$ is a function of $y'$, $P$ must also be a function of $y'$; so that (B) may be written

$$f(y') + y'F(y') = c = f'(y'),$$

where $f'$ is an arbitrary function. The last equation can therefore only be satisfied by making $y'$ a constant, say $y' = c_1$, which gives $y = c_1 x + c_2$.

Hence if we require the nature of the curve which will maximize or minimize the expression $U=\int_{x_0}^{x_1} Vdx$, where $V$ is any function of $y'$ only, the straight line is the solution, if there be a solution; that question being decided by an appeal to the terms of the second order.

*Second.* If $V$ be a function of $y$ and $y'$ only, (A) will still give

$$V=c+Py'. \qquad \text{(C)}$$

*Third.* If $V$ be a function of $y$ and $y''$ only, then (A) will give

$$V=c+Qy''-\frac{dQ}{dx}y'. \qquad \text{(D)}$$

**57.** Suppose, in the third place, that the independent variable $x$, and also the first $m$ of the quantities $y$, $y'$, $y''$, etc., are wanting in $V$; then the equation $M=0$ can be integrated at least $m+1$ times. Let $m$, for example, be 4 as formerly. Then the equation $M=0$, after having been integrated four times, according to the first case, and using $p$, $q$, $r$, $s$, etc., for $P$, $Q$, $R$, $S$, etc., to prevent confusion, becomes

$$s-\frac{dt}{dx}+\frac{d^2u}{dx^2}-\text{etc.}=ax^3+bx^2+cx+d. \qquad (1)$$

Also, we have the equation

$$\begin{aligned} dV &= sdy^{\text{iv}}+tdy^{(5)}+udy^{(6)}+\text{etc.} \\ &= (sy^{(5)}+ty^{(6)}+uy^{(7)}+vy^{(8)}+\text{etc.})\,dx. \qquad (2)\end{aligned}$$

Substituting in (2) the value of $s$ derived from (1), we have

$$dV=\left(ty^{(6)}+\frac{dt}{dx}y^{(5)}\right)dx+\left(uy^{(7)}-\frac{d^2u}{dx^2}y^{(5)}\right)dx$$

$$+\left(vy^{(7)}+\frac{d^3v}{dx^3}y^{(5)}\right)dx+\text{etc.}+(ax^3+bx^2+cx+d)\,y^{(5)}\,dx. \qquad (3)$$

Integrating by parts, as in the second case, we have

$$V = e + ty^{(5)} + \left(uy^{(6)} - \frac{du}{dx}y^{(5)}\right) + \left(vy^{(7)} - \frac{dv}{dx}y^{(6)} + \frac{d^2v}{dx^2}y^{(5)}\right) + \text{etc.}$$

$$+\int (ax^3 + bx^2 + cx + d)\,y^{(5)}\,dx. \qquad (4)$$

Moreover, the integral sign can easily be removed from the remaining terms in (4). For, by parts, we have

$$\int ax^3y^{(5)}dx = ax^3y^{\text{iv}} - \int 3ax^2y^{\text{iv}}dx,$$

$$-\int 3ax^2y^{\text{iv}}dx = -3ax^2y''' + \int 6axy'''dx,$$

$$\int 6axy'''dx = 6axy'' - \int 6ay''dx,$$

$$-\int 6ay''dx = -6ay'.$$

Hence

$$\int ax^3y^{(5)}dx = ax^3y^{\text{iv}} - 3ax^2y''' + 6axy'' - 6ay';$$

and in like manner we may integrate all the other terms.

Thus, for example, in Prob. IV. we find, after two integrations of $M$,

$$Q \text{ or } \frac{dV}{dy''} = cx + c',$$

which, being again integrated, gives

$$V = cxy'' - cy' + c'y'' + d.$$

Or, let $V$ be a function of $y'$ and $y''$ only. Then, after one integration of $M$, we have

$$P-\frac{dQ}{dx}=a.$$

We also have

$$dV=Pdy'+Qdy'';$$

and substituting the value of $P$ from the preceding equation, we have

$$dV=\left(ay''+Qy'''+\frac{dQ}{dx}y''\right)dx,$$

which, being integrated, gives

$$V=ay'+Qy''+b.$$

### Problem VI.

**58.** *It is required to determine the form of the solid of revolution which will experience a minimum resistance in passing through a homogeneous fluid in the direction of $x$, the axis of revolution of the solid.*

Although it is evident that the problem does not admit of a solution until some further restrictions are imposed, we shall at present merely assume that the distance $x_1 - x_0$ is given.

Let $ds$ be an element of the generating curve, $pds$ the normal pressure which it experiences in passing through the fluid, and $v$ its velocity in the direction of that normal, or the velocity with which the particles of the fluid are displaced by it in that direction. Then, adopting the usual theory regarding the pressure and resistance of fluids, we have

$$pds=cv^2ds, \tag{1}$$

where $c$ is a constant depending upon the density of the fluid. Let $v'$ be the velocity of the body in the direction of the axis $x$. Then

$$v = v' \frac{dy}{ds}, \tag{2}$$

and (1) becomes

$$pds = cv'^2 \frac{dy^2}{ds^2} ds. \tag{3}$$

Let $dz$ be the surface of the elemental zone, described by $ds$. Then, since $dz = 2\pi yds$, we shall have $pdz$, or the normal pressure upon this zone,

$$= 2\pi ypds = 2\pi cv'^2 \frac{ydy^2}{ds^2} ds. \tag{4}$$

Now the force $pdz$, being distributed normally about the zone, may be regarded as aggregated and exerted in the direction of any particular normal, and may, moreover, be resolved into two components, the first in the direction of $x$, and the second in the direction of $y$, and the first is the only one which resists the motion of the body in the direction of $x$. Therefore, $rdz$ being the resistance of any elemental zone to the forward motion of the body, we have

$$rdz = pdz \frac{dy}{ds} = 2\pi cv'^2 \frac{ydy^3}{ds^3} ds, \tag{5}$$

in which only the numerical values of the quantities have been regarded. Let $R$ be the resistance of any zone included between the two planes $x = x_0$, $x = x_1$, and we shall have

$$R = 2\pi cv'^2 \int_{s_0}^{s_1} \frac{ydy^3}{ds^3} ds = 2\pi cv'^2 \int_{x_0}^{x_1} \frac{yy'^3}{1 + y'^2} dx.$$

Now we shall not regard the resistance experienced by any plane cylindrical end, should there be one, so that $R$ is the

quantity which must become a minimum. Therefore, neglecting, as usual, the constants $c$, $\pi$, $v'$, we are to minimize the expression

$$U = \int_{x_0}^{x_1} \frac{yy'^2}{1+y'^2}\,dx = \int_{x_0}^{x_1} V\,dx.$$

Here, as $V$ is a function of $y$ and $y'$ only, and as

$$P = y\,\frac{3y'^2(1+y'^2) - 2y'^4}{(1+y'^2)^2},$$

we have, by formula (C), Art. 56,

$$\frac{yy'^2}{1+y'^2} = yy'\,\frac{3y'^2(1+y'^2) - 2y'^4}{(1+y'^2)^2} - 2c, \tag{6}$$

which is the convenient form of the constant. Whence we derive the equations

$$\frac{yy'^2}{1+y'^2} = \frac{3yy'^2}{1+y'^2} - \frac{2yy'^4}{(1+y'^2)^2} - 2c,$$

$$\frac{yy'^2}{1+y'^2} - \frac{yy'^4}{(1+y'^2)^2} = c. \tag{7}$$

Reducing the first member of (7) to a common denominator and solving for $y$, we obtain

$$y = \frac{c(1+y'^2)^2}{y'^2}. \tag{8}$$

From this differential equation, although it cannot be further integrated, we may obtain the value of $x$. For differentiate (8), and we have

$$dy \text{ or } y'dx = c\,\frac{y'^2\,4y'(1+y'^2)dy' - (1+y'^2)^2 3y'^2 dy'}{y'^6}. \tag{9}$$

Whence

$$dx = c\frac{4y'^2(1+y'^2)-3(1+y'^2)^2}{y'^5}dy'$$

$$= c\frac{y'^4-2y'^2-3}{y'^5}dy'$$

$$= c\left(\frac{dy'}{y'}-\frac{2dy'}{y'^3}-\frac{3dy'}{y'^5}\right), \tag{10}$$

which is easily integrated, giving

$$x = c\left(ly'+\frac{1}{y'^2}+\frac{3}{4y'^4}\right)+d. \tag{11}$$

Now if we suppose equations (8) and (11) to be combined so as to eliminate $y'$, we shall obtain an equation between $x$, $y$, $c$, and $d$, which will be the equation of the required curve in finite terms, and may be supposed to be expressed under the form

$$f(x, y, c, d) = 0 = f.$$

Then if we suppose the values of $y_1$ and $y_0$ to be given, we shall have

$$f(x_1, y_1, c, d) = 0, \qquad f(x_0, y_0, c, d) = 0;$$

from which equations we must determine $c$ and $d$ in terms of the given quantities $x_0$, $x_1$, $y_0$, $y_1$. But if $y_1$ and $y_0$ be not given, we shall have

$$h_1 = P_1 = y_1\left(\frac{3y'^2(1+y'^2)-2y'^4}{(1+y'^2)^2}\right)_1 = 0;$$

which gives either $y_1 = 0$ or $y_1' = 0$; and a similar equation for the lower limit.

But it is easy to see that the form of $f$ cannot be practically determined, as the elimination of $y'$ just proposed can-

not be effected; while it is well known that the theory which we have adopted regarding the resistance of fluids is not altogether trustworthy. The problem will, however, afford ground hereafter for some useful remarks regarding the terms of the second order, and is also of historic interest, having occupied the attention of Newton, Legendre, and others.

## Problem VII.

**59.** *It is required to determine among all curves which can be drawn between two fixed points, that which, being revolved about the axis of $x$, will generate the surface of minimum area.*

Let $ds$ be an element of the generating curve. Then the surface which is to become a minimum will be $2\pi\int_{s_0}^{s_1} y\,ds$, or $2\pi\int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx$, so that, neglecting the constant, we must minimize the expression

$$U = \int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx = \int_{x_0}^{x_1} V\,dx.$$

Here $V$ is a function of $y$ and $y'$ only, and $P = \dfrac{yy'}{\sqrt{1+y'^2}}$. Hence, by formula (C), Art. 56, we have the equations

$$y(1+y'^2) = y'\frac{yy'}{\sqrt{1+y'^2}} + a, \tag{1}$$

$$\frac{y}{\sqrt{1+y'^2}} = a. \tag{2}$$

Squaring, clearing fractions, and transposing, we have

$$y'^2 = \frac{y^2 - a^2}{a^2}. \tag{3}$$

To render (3) integrable, we must solve thus:

$$dx = \pm \frac{a\,dy}{\sqrt{y^2 - a^2}}, \tag{4}$$

the integral of which, using the upper sign, is

$$x = al\left(y + \sqrt{y^2 - a^2}\right) + b, \tag{5}$$

the equation of the catenary, as we will next show.

Now it is plain that if we regard the axis of $x$ as fixed, but that of $y$ as movable, we can render $b$ any quantity we please by suitably choosing the position of that movable axis; that is, by suitably determining the origin on the fixed axis $x$. In this case let it be so taken that $b$ becomes $-ala$, and then (5) will become

$$x = al\frac{y + \sqrt{y^2 - a^2}}{a}. \tag{6}$$

Let $e$ be the Napierian base, then (6) will give

$$ae^{\frac{x}{a}} = y + \sqrt{y^2 - a^2}. \tag{7}$$

But from (3) we obtain

$$\sqrt{y^2 - a^2} = ay',$$

whence (7) gives

$$ae^{\frac{x}{a}} = y + ay'. \tag{8}$$

Now if in (7) we make $y = 0$, $x$ becomes imaginary. Whence the curve does not meet the axis of $x$, and $y$ is always positive. But if we make $x = 0$, $y$ becomes $a$, and $y'$ at this point becomes zero, it being zero at no other. Moreover, we have $y'' = \frac{y}{a^2}$, so that the curve is convex to the axis of $x$, and is without cusps or points of inflection. Therefore $y'$ changes sign when $x = 0$, and also we have certainly a minimum ordi-

nate at that point. Now as points which have equal ordinates have also $y'$ numerically equal, but positive or negative according as the point lies at the right or left of the origin, and as (6) shows that there can be no two equal values of $y$ on the same side of the origin, we conclude that the curve has—at least so far as it extends—for every point at the right of the origin, a point at the left, having an equal ordinate, while the values of $x$ and $y'$ are numerically equal, but with contrary signs. Hence we may also write

$$ae^{-\frac{x}{a}} = y - ay'. \tag{9}$$

Therefore, adding (8) and (9), we obtain

$$y = \frac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \tag{10}$$

which is the usual form of the equation to the catenary when the directrix is the axis of $x$, and the origin under the lowest point; also, $a$ is the constant which would in mechanics represent the tension in the direction of $x$.

**60.** We have already seen how to dispose of the constant $b$ which occurs in the general solution, and we now proceed to consider the remaining constant $a$.

It must be evident that even when the limiting values of $x$ and $y$ are given—just as when they are not—it may happen that no constants can satisfy the given conditions; that is, that no curve of the required kind can be drawn between the given points. Let us first suppose that the two points are equally distant from the axis of $x$, and let $x_1 = c$ and $y_1 = b$. Then (10) gives

$$b = \frac{1}{2}a\left(e^{\frac{c}{a}} + e^{-\frac{c}{a}}\right), \tag{11}$$

and from this equation $a$ must be found in terms of $c$ and $b$.

But we are chiefly concerned in knowing when, if at all, the solution will become impossible; and this point we will now consider. If we differentiate the second member of (11) under the supposition that $c$ is constant and $a$ variable, and then equate the result to zero, we shall obtain, on solving, the values of $a$ expressed in terms of $c$, if any exist, which will render $b$ a minimum. Performing this operation, we have

$$\frac{1}{2}\left(e^{\frac{c}{a}}+e^{-\frac{c}{a}}\right)-\frac{c}{2a}\left(e^{\frac{c}{a}}-e^{-\frac{c}{a}}\right)=0. \qquad (12)$$

Developing each term of (12) carefully by Maclaurin's Theorem, we have

$$1-\frac{1}{2}\frac{c^2}{a^2}-\frac{3}{\underline{|4}}\frac{c^4}{a^4}-\text{etc.}=0, \qquad (13)$$

an equation which evidently gives but one positive value for $a$, because its first member is $-\infty$ when $a$ is zero, and unity when $a$ is infinite. But (12), when solved for $\frac{c}{a}$, is known to give approximately $\frac{c}{a}=1.19968=d$, which evidently renders $b$ a minimum, as it is clear from (11) that we can make it infinite by making $a$ infinite. To determine this minimum value of $b$ in terms of $c$, first substitute in (11) thus:

$$b=\frac{a}{2}\left(e^d+\frac{1}{e^d}\right),$$

which, being solved, is known to give approximately $\frac{b}{a}=1.81017$. Therefore we have $\frac{b}{c}=1.5088$.

Now as this equation gives the least value of $b$ in terms of $c$, it is evident that if the extreme points be so given that $\frac{b}{c}$ will be less than 1.5088, there can be no catenary drawn hav-

ing the axis of $x$ as its directrix, although of course some catenary can always be drawn; and if $\frac{b}{c}$ become equal to 1.5088, then a single catenary can be drawn in which $a$ must equal $\frac{c}{1.19968}$ or $\frac{b}{1.81017}$; and if $\frac{b}{c}$ become greater than 1.5088, then two real and positive values can be found for $a$, and we may, by using each in succession, draw two catenaries between the two given points, each having the axis of $x$ as its directrix.*

**61.** As it will be found highly important, in determining the question of their minimum property, to distinguish between the upper and lower catenary, we must now also consider the more general case in which $y_0$ and $y_1$ are unequal.

Suppose, then, that the given points are unequally distant from the axis of $x$, $x$ being so estimated that $y_1$ shall be greater than $y_0$. Then move the origin along the axis of $x$ to a point midway between the ordinates $y_0$ and $y_1$. Denote $y_1$ by $b$, $y_0$ by $k$, $x_1$ by $c$, and $x_0$ by $-c$. Then $n$ being the distance of the new origin from the former, and of course positive, the general equation of the catenary becomes

$$y=\frac{a}{2}\left(e^{\frac{x+n}{a}}+e^{-\frac{x+n}{a}}\right). \tag{1}$$

Hence we have at the limits the equations

$$b=\frac{a}{2}\left(e^{\frac{c+n}{a}}+e^{-\frac{c+n}{a}}\right), \tag{2}$$

$$k=\frac{a}{2}\left(e^{\frac{n-c}{a}}+e^{-\frac{n-c}{a}}\right). \tag{3}$$

From these equations we must now find $a$ and $n$. Mul-

tiply (2) by $e^{\frac{c}{a}}$ and (3) by $e^{-\frac{c}{a}}$, and subtract; then multiply (2) by $e^{-\frac{c}{a}}$, and (3) by $e^{\frac{c}{a}}$, and subtract. Then we shall have the two equations

$$\frac{a}{2}e^{\frac{n}{a}}\left(e^{\frac{2c}{a}}-e^{-\frac{2c}{a}}\right)=be^{\frac{c}{a}}-ke^{-\frac{c}{a}}, \tag{4}$$

$$\frac{a}{2}e^{-\frac{n}{a}}\left(e^{-\frac{2c}{a}}-e^{\frac{2c}{a}}\right)=be^{-\frac{c}{a}}-ke^{\frac{c}{a}}. \tag{5}$$

Changing signs in (5) and multiplying by (4), we have

$$\frac{a^2}{4}\left(e^{\frac{2c}{a}}-e^{-\frac{2c}{a}}\right)^2=\left(be^{\frac{c}{a}}-ke^{-\frac{c}{a}}\right)\left(ke^{\frac{c}{a}}-be^{-\frac{c}{a}}\right). \tag{6}$$

Having thus eliminated $n$, we must now determine whether (6) can be satisfied by any real and positive value or values of $a$. Write, for convenience,

$$F=\frac{a^2}{4}\left(e^{\frac{2c}{a}}-e^{-\frac{2c}{a}}\right)^2-\left(be^{\frac{c}{a}}-ke^{-\frac{c}{a}}\right)\left(ke^{\frac{c}{a}}-be^{-\frac{c}{a}}\right), \tag{7}$$

which becomes zero whenever a catenary is possible. Differentiating $F$ under the supposition that $a$ only is variable, we obtain

$$\frac{dF}{da}=F'=\left\{e^{\frac{2c}{a}}-e^{-\frac{2c}{a}}\right\}$$

$$\left\{\frac{a}{2}\left(e^{\frac{2c}{a}}-e^{-\frac{2c}{a}}\right)-c\left(e^{\frac{2c}{a}}+e^{-\frac{2c}{a}}\right)+\frac{2cbk}{a^2}\right\}. \tag{8}$$

Now if $F'$ can vanish for any real and positive value of $a$, $F$ has a corresponding minimum value or values. For it has its greatest when $a$ is zero, its value then being infinite. For

if we develop by Maclaurin's Theorem the first member of the following equation, we shall have

$$\frac{a}{2}\left(e^{\frac{2c}{a}} - e^{-\frac{2c}{a}}\right) = 2c\left(1 + \frac{2^2c^2}{a^2}\frac{1}{\lfloor 3} + \frac{2^4c^4}{a^4}\frac{1}{\lfloor 5} + \text{etc.}\right).$$

Whence, when $a$ is infinite, $F$ becomes $4c^2 + (b - k)^2$; and when $a$ is zero, $F$ becomes infinite.

Now to determine whether $F'$ can vanish and change its sign as $a$ ranges from zero to positive infinity, we must recollect that $e^{\frac{2c}{a}} - e^{-\frac{2c}{a}}$ is of invariable sign, and that therefore the only part of $F'$ which can change its sign is the second factor, and this, when developed by Maclaurin's Theorem and arranged, becomes

$$2c\left\{\frac{bk - \frac{4c^2}{3}}{a^2} - \frac{2^4c^4}{a^4}\left(\frac{1}{\lfloor 4} - \frac{1}{\lfloor 5}\right) - \text{etc.}\right\}. \qquad (9)$$

Now if $bk$ be greater than $\frac{4c^2}{3}$, we can evidently make (9), and consequently $F'$, vanish and change its sign once, and once only, for any real and positive value of $a$; and therefore $F$ is in this case susceptible of a minimum value; and if this minimum value be negative, $F$ can be made to pass through zero, and to change its sign twice. Hence in this case equation (7), or $F = 0$, can be satisfied by two real values of $a$; and we can draw two catenaries by using these values successively. But if the minimum value of $F$ be zero, $F$ can touch zero but once, and (7) can only be satisfied by one value of $a$, and thus we can draw but one catenary. If the minimum value of $F$ be positive, then $F$ cannot become zero at all, and (7) cannot be satisfied by any real and positive value of $a$, and thus no catenary can be drawn.

Now if $bk$ be equal to or less than $\frac{4c^2}{3}$, $F'$ will be always negative, and $F$ can consequently have no minimum value. In this case $F$ cannot touch zero at all, and there can be no catenary. For we have already seen that the least value of $F$ is $4c^2+(b-k)^2$, which expression is evidently greater than zero.

**62.** The preceding article is taken from Chapter IV. of the Adams Essay, by Prof. Todhunter; and we shall now, before closing this section, subjoin an investigation of the terms of the second order in a particular class of problems, in which Probs. VI. and VII. are included. This investigation appears to be due entirely to the same author. (See Adams Essay, Arts. 26, 27.)

### Problem VIII.

*It is required to investigate in full the conditions which will maximize or minimize the expression*

$$U=\int_{x_0}^{x_1} yf dx=\int_{x_0}^{x_1} V dx,$$

*where $f$ is any function of $y'$ only.*

Put $f'$ for $\frac{df}{dy'}$, and $f''$ for $\frac{d^2f}{dy'^2}$. Then, to the terms of the second order inclusive, we shall have

$$\delta U=\int_{x_0}^{x_1}(f\delta y+yf'\delta y')dx$$

$$+\frac{1}{2}\int_{x_0}^{x_1}(2f'\delta y\,\delta y'+yf''\delta y'^2)dx. \qquad (1)$$

Here $V$ is a function of $y$ and $y'$ only, and $P=yf'$, so that we have at once, by formula (C), Art. 56, since the terms of the first order must vanish,

$$yf-yy'f'=c. \tag{2}$$

This is as far as we can carry the general solution, so long as the form of $f$ is entirely arbitrary, although we may suppose the solution to be of the form

$$y=F(x, c, c').$$

**63.** Let us now consider what transformations can be effected in the terms of the second order. By parts we have

$$\int f'\delta y\,\delta y' dx = f'\delta y^2 - \int \delta y \,.\, \frac{d}{dx} f'\delta y.dx. \tag{3}$$

But

$$\frac{d}{dx} f'\delta y = f'\delta y' + \delta y \frac{df'}{dx}.$$

Whence

$$\int f'\delta y\,\delta y' dx = f'\delta y^2 - \int f'\,\delta y\,\delta y' dx - \int \frac{df'}{dx}\,\delta y^2 dx. \tag{4}$$

Therefore

$$2\int f'\delta y\,\delta y' dx = f'\delta y^2 - \int \frac{df'}{dx}\,\delta y^2 dx. \tag{5}$$

Substituting this value in (1), and observing that the terms of the first order vanish, and that $\dfrac{df'}{dx} = f''y''$, we have

$$\delta U = \frac{1}{2}\left(f_1'\delta y_1^2 - f_0'\delta y_0^2\right) + \frac{1}{2}\int_{x_0}^{x_1}\left(yf''\delta y'^2 - y''f''\delta y^2\right)dx. \tag{6}$$

But if we suppose, as usual, the values of $y_0$ and $y_1$ to be assigned, we have

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} f''(y\delta y'^2 - y''\delta y^2)dx. \tag{7}$$

Now in our applications of this formula we shall usually be able to regard $y$ as positive; and let us also suppose that $y''$ does not change its sign within the range of integration; that is, that the required curve is always, at least for the part that we consider, convex or concave to the axis of $x$. We will consider first the latter case. Here, since $y''$ is always negative, the factor $y\delta y'^2 - y''\delta y^2$ is always positive, and therefore, if $f''$ be also of invariable sign, we shall have a maximum or a minimum acording as it is negative or positive.

But we can show that when $y''$ is of invariable sign, $f''$ must be also. For from (2) we have

$$y = \frac{c}{f - y'f'}. \tag{8}$$

Whence, by differentiation, we have

$$y' = \frac{cy'y''f''}{(f - y'f')^2}; \tag{9}$$

and solving for $y''$, we find

$$y'' = \frac{(f - y'f')^2}{cf''}; \tag{10}$$

which shows that when $y''$ is of invariable sign, $f''$ must be also. But since $c$ may be either positive or negative, the sign of $f''$ cannot be determined so long as the problem is perfectly general, and therefore we can only say that when $y''$ is negative, we have a maximum or minimum according as $f''$ is negative or positive.

Next suppose $y''$ to be always positive. Then, although $f''$ must be of invariable sign, we cannot say that the factor $y\delta y'^2 - y''\delta y^2$ may not change its sign; and therefore this case will require further investigation, which will be given hereafter, when we have presented Jacobi's Theorem.

**64.** We will now apply the preceding formula to the investigation of the terms of the second order in Prob. VI., although we did not succeed in obtaining the equation of the generating curve in finite terms.

From equation (9), Art. 58, we easily obtain

$$y' = c(1 + y'^2)\frac{(y'^2 - 3)y''}{y'^4}. \tag{1}$$

Now we will consider the case in which $y'$ is positive and of invariable sign. Then, observing that $\tan^2 60° = 3$, we see that if $y'^2$ be always less than 3, $y''$ will be always negative; if $y'^2$ pass through 3, $y''$ will change its sign; and if $y'^2$ be always greater than 3, $y''$ will be always positive. Hence we may investigate the first case. Write the fundamental equation of Art. 58 thus:

$$U = \int_{x_0}^{x_1} \frac{yy'^3}{(1 + y'^2)}\,dx = \int_{x_0}^{x_1} yf dx.$$

Then we find

$$f' = \frac{3y'^2 + y'^4}{(1 + y'^2)^2},$$

and

$$f'' = 2y'\frac{3 - y'^2}{(1 + y'^2)^3}.$$

Hence, if we suppose the limiting values of $x$ and $y$ to be fixed, the terms of the second order will become

$$\int_{x_0}^{x_1} \frac{y'}{(1 + y'^2)^3}(3 - y'^2)\,(y\delta y'^2 - y''\delta y^2)dx,$$

which expression is evidently positive, thus giving us a minimum.

**65.** The term minimum must here also be understood in its technical sense, and we must by no means say that the curve whose differential equation we have obtained is the curve which will generate the solid of least resistance. For Legendre has shown that by taking a zigzag line we can make the resistance as small as we please. The fact is that in this case, as in every other, we can, by means of the calculus of variations, compare the curve or curves obtained from the differential equation $M = 0$ with such curves only as can be derived from their primitives by making infinitesimal changes in the values of $y$ and $y'$. And although we might pass from a continuous curve to a zigzag line by means of infinitesimal changes in $y$, we certainly could not by such changes in $y'$.

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## SECTION III.

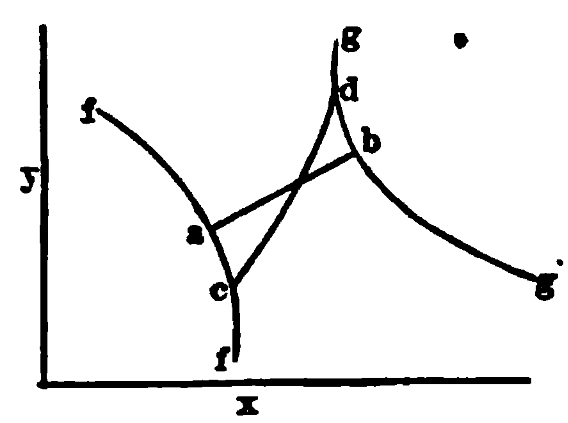
### *CASE IN WHICH THE LIMITING VALUES OF X ALSO ARE VARIABLE.*

### Problem IX.

**66.** *Suppose it be required to find the shortest curve which can be drawn so as to connect two given curves, all the curves lying in the same plane.*

Let $ff$ and $gg$ be the given curves, and $ab$ the required curve, which is of course a right line. Then if we assume two other points, $c$ and $d$, indefinitely near to $a$ and $b$, and join them by another curve which is of the same kind as, or differs infinitesimally in form from, $ab$, the curve $cd$ must exceed in length the curve $ab$.

This assertion would be equally true if the points $a$ and $c$, $b$ and $d$ had not been taken indefinitely near, and if the curve $cd$ had not differed infinitesimally in form from $ab$. But then, even if $cd$ were shown to exceed $ab$ in length, we could not be certain that some third curve might not be drawn be-




tween $ff$ and $gg$ differing less in form from $ab$, or having its extremities a little nearer to $a$ and $b$, which might be shorter than either $ab$ or $cd$.

Now since, whatever be its nature, the length of the line $ab$ is given by the equation

$$U = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx = \int_{x_0}^{x_1} V dx,$$

we see that we are now required to determine what change $U$ undergoes when not only $y'$, but also the co-ordinates $x_0, y_0, x_1, y_1$, of the points $a$ and $b$, receive indefinitely small increments.

Now it is evident that we may pass from the curve $ab$ to the curve $cd$ in the following manner: First, without at all changing the form of $ab$, give indefinitely small increments $dx_0$ and $dx_1$ to the original limits $x_0$ and $x_1$, so that the new values of these limits may become the abscissæ of the new points $c$ and $d$, which change would give to the curve $ab$ an increment like that which it would receive by differentiation. Then, secondly, vary $y'$, supposing $x$ throughout the new limits to be incapable of variation. By the change in the

limits we obtain the required abscissæ of the new extreme points, while by varying $y'$ we obtain their ordinates, and also any desired alteration in the form of the primitive curve.

Now if we denote by $U'$ what $U$ becomes when we change $x_0$ into $x_0 + dx_0$, and $x_1$ into $x_1 + dx_1$, the form of the primitive curve being unchanged—that is, $y$ and $y'$ being unvaried—we shall have

$$U' = \int_{x_0+dx_0}^{x_1+dx_1} Vdx$$

$$= \int_{x_0}^{x_1} Vdx + \int_{x_1}^{x_1+dx_1} Vdx - \int_{x_0}^{x_0+dx_0} Vdx, \qquad (1)$$

in which expression it must be remembered that the increments $dx_0$ and $dx_1$ are to be taken either positively or negatively, according as the abscissæ of the new extreme points are further from, or nearer to $-\infty$ than those of the original points. But it is evident that to the second order

$$\int_{x_1}^{x_1+dx_1} Vdx = V_1 dx_1 + \frac{1}{2}\left(\frac{dV}{dx}\right)_1 dx_1^2; \qquad (2)$$

and making the same reduction for the lower limit, (1) becomes

$$U' = V_1 dx_1 - V_0 dx_0 + \frac{1}{2}\left\{\left(\frac{dV}{dx}\right)_1 dx_1^2 - \left(\frac{dV}{dx}\right)_0 dx_0^2\right\}$$

$$+ \int_{x_0}^{x_1} Vdx, \qquad (3)$$

which is true to the second order.

Let us next ascertain, as far as the terms of the second order, what change would result to $U'$ from changing $y'$ into $y' + \delta y'$. Since the integral in (3) equals $U$, the change which will result to it will be merely $\delta U$, where $\delta U$ is to be found to the

second order, and the terms of the first order transformed as hitherto explained, so that we need only consider the terms without the integral sign. The change in the term $V_1 dx_1$, produced by varying $y_1'$, $dx_1$ remaining unaltered, is $\delta V_1 dx_1$, which, if we put as usual $P$ for $\frac{y'}{\sqrt{1+y'^2}}$, becomes

$$P_1\,\delta y_1'\,dx_1 + \left(\frac{1}{2V^3}\right)_1 \delta y_1'^2 dx_1,$$

the first term only being retained, as the others are evidently of an order higher than the second; and similarly for the corresponding term at the lower limit. The term $\frac{1}{2}\left(\frac{dV}{dx}\right)_1 dx_1^2$ is already of the second order, and must simply be retained without regarding its variation, every term of which would be of an order higher than the second. Similarly, we merely retain the corresponding term at the lower limit.

Now collecting our results, and denoting by $[\delta U]$ the entire change which the length of *ab* or $U$ has undergone, we shall have

$$[\delta U] = V_1\,dx_1 - V_0\,dx_0 + P_1\,\delta y_1 - P_0\,\delta y_0 + \int_{x_0}^{x_1} - \frac{dP}{dx}\,\delta y\,dx$$

$$+ \frac{1}{2}\left\{\left(\frac{dV}{dx}\right)_1 dx_1^2 - \left(\frac{dV}{dx}\right)_0 dx_0^2\right\} + P_1\,\delta y_1' dx_1 - P_0\,\delta y_0' dx_0$$

$$+ \frac{1}{2}\int_{x_0}^{x_1} \frac{1}{V^3}\,\delta y'^2 dx. \qquad (4)$$

**67.** Now it will appear, by reasoning in all respects similar to that which has been hitherto employed, that since $dx_0$ and $dx_1$, like $\delta y_0$ and $\delta y_1$, are capable of either sign, if $U$ is to be a maximum or minimum, the terms of the first order in $[\delta U]$ must vanish, and those of the second must become positive

for a minimum and negative for a maximum. Disregarding, therefore, at present the terms of the second order, we have

$$[\delta U] = V_1 dx_1 - V_0 dx_0 + P_1 \delta y_1 - P_0 \delta y_0 + \int_{x_0}^{x_1} - \frac{dP}{dx} \delta y \, dx. \quad (5)$$

Now it must be evident that the curve sought can be no other than a straight line. For suppose the points $a$ and $b$ to be joined by any curve other than a straight line. Then even if this curve were shorter than any other line which could be drawn between the given curves, when one or both the extreme points $a$ and $b$ were changed, yet we know from our previous investigations that, without changing these points, it could be still further shortened by making it a right line. Whence we see that our present problem must concern merely the position which this line must have in order to render its length a minimum. Moreover, the term under the integral sign in (5) is just what it would have been had we merely required the curve of minimum length between two fixed points. Therefore, since the right line is the general solution, $\frac{dP}{dx}$ will vanish, and consequently the integral must vanish, thus leaving us with the terms at the limits, which must also be equated to zero.

This mode of demonstration will probably be most apparent, but the following is the true analytical method. By reasoning similar to that employed in Art. 39 and the preceding articles, we can show that the term under the integral sign must vanish, as must also those free from the sign of integration, taken collectively. Equating the integral to zero, we obtain, as before, the right line as the general solution, and have then to consider the remaining terms, which may be represented by the equation $L = 0$.

**68.** We have, then, from (5),

$$L = V_1 dx_1 - V_0 dx_0 + P_1 \delta y_1 - P_0 \delta y_0 = 0. \quad (6)$$

Now if the quantities $dx_1$, $dx_0$, $\delta y_1$, $\delta y_0$ were entirely independent, we would evidently be obliged to equate the coefficient of each one severally to zero. Then we would have four equations at the limits to be satisfied, whereas the general solution contains but two arbitrary constants, and this would usually be impossible in any problem. But in the present case we know, without further investigation, that two of these equations, $V_1 = 0$ and $V_0 = 0$, cannot be satisfied by any real value of $y'$. This is as it should be. For if the quantities $dx_1$, $dx_0$, $\delta y_1$, $\delta y_0$ were independent, the extremities of the required curve would be entirely unrestricted, and we could have no maximum or minimum, because we could always increase or diminish its length at pleasure. But as in the present case the extremities of the required curve are confined to two given curves, we can obtain a definite result, and we now proceed to show the method of imposing this condition upon the question.

**69.** Let $y = f(x)$ and $y = F(x)$ be the equations of the two given curves, and let $y$ be any ordinate of the required curve, and $Y$ the ordinate of the derived curve corresponding to the same value of $x$. Then $Y_0 = y_0 + \delta y_0$, and $Y_1 = y_1 + \delta y_1$; and let us consider, for example, the upper limit. It is evident that when we derive $cd$ from $ab$, the abscissa of $d$ will become $x_1 + dx_1$, where $dx_1$ may be positive or negative, and the value of its ordinate is evidently obtained by passing along the derived curve from the point whose co-ordinates are $x_1$ and $Y_1$ to the point whose abscissa is $x_1 + dx_1$; that is, to the point $d$. Denoting then the ordinate of $d$ by $n$, we have

$$n = Y_1 + Y_1' dx_1 + \frac{1}{2} Y_1'' dx_1^2 + \text{etc.} \tag{7}$$

Hence, substituting in (7) the value $Y_1 = y_1 + \delta y_1$, and omitting all terms of an order higher than the second, we have

$$n = \left(y + \delta y + y'dx + \delta y'dx + \frac{1}{2}y''dx^2\right)_1. \qquad (8)$$

But since $n$ is an ordinate of the given curve whose equation is $Y = f(x)$, we must have $n = f(x_1 + dx_1)$. Developing this expression by Taylor's Theorem, we have to the second order

$$n = f_1 + f_1'dx_1 + \frac{1}{2}f_1''dx_1^2, \qquad (9)$$

where

$$f' = \frac{df}{dx}, \qquad f'' = \frac{d^2f}{dx^2}, \qquad f_1 = f(x_1).$$

Combining (8) and (9), observing that $y_1 = f_1$, we have

$$\delta y_1 = (f' - y')_1 dx_1 + \frac{1}{2}(f'' - y'')_1 dx_1^2 - \delta y_1' dx_1. \qquad (10)$$

Similarly, we have at the lower limit

$$\delta y_0 = (F' - y')_0 dx_0 + \frac{1}{2}(F'' - y'')_0 dx_0^2 - \delta y_0' dx_0. \qquad (10)$$

**70.** If now we substitute in (6) the values of $\delta y_1$ and $\delta y_0$ just found, and set aside all terms of the second order, which must be added to those of the second order in (4), we shall, after restoring the values of $V$ and $P$, have

$$L = \left\{\frac{y'f' - y'^2}{\sqrt{1 + y'^2}} + \sqrt{1 + y'^2}\right\}_1 dx_1$$

$$- \left\{\frac{y'F' - y'^2}{\sqrt{1 + y'^2}} + \sqrt{1 + y'^2}\right\}_0 dx_0 = 0. \qquad (11)$$

Having thus eliminated $\delta y_1$ and $\delta y_0$, it is evident that the remaining quantities $dx_1$ and $dx_0$ are absolutely independent, and that we must therefore equate their coefficients severally

to zero. Performing this operation, and reducing, we have the equations

$$1 + y_1'f_1' = 0, \qquad \text{and} \quad 1 + y_0'F_0' = 0;$$

equations which show that the required right line $ab$ must be normal to each of the curves $ff$ and $gg$.

**71.** Although for the sake of simplicity we have used equation (6), it is evident that the true mode of reasoning would be the following: First eliminate $\delta y_1$ and $\delta y_0$ in (4) by the use of equation (10), by which elimination we shall add some terms to those of the second order. Then, by the usual reasoning, those of the first order must vanish. But these terms will then consist of $L$ as given in (11), together with an integral; and, by the reasoning already employed, these two parts must separately vanish. Now by making the integral vanish, we obtain the right line as the general solution; while by making $L$ vanish, we obtain at once equation (11), from which we derive the same conclusion as before.

**72.** If we use equation (6), recollecting that it is true to the first order only, we may evidently obtain the complete terms of the second order by adding to those already in (4) those which result from the elimination of $\delta y_1$ and $\delta y_0$ in (6) by the use of equation (10). But these terms will by either method become, since those of the first order vanish,

$$[\delta U] = \frac{1}{2}\left\{\left(\frac{y'f''}{\sqrt{1+y'^2}}\right)_1 dx_1^2 - \left(\frac{y'F''}{\sqrt{1+y'^2}}\right)_0 dx_0^2\right\}$$

$$+\int_{x_0}^{x_1} \frac{1}{2(1+y'^2)^{\frac{3}{2}}}\,\delta y'^2 dx. \tag{12}$$

Now the integral in (12) is known from Prob. I. to be positive, so that we shall be sure of a minimum if the remaining

terms be positive, but not otherwise. But since the solution is a right line, $\frac{y'}{\sqrt{1+y'^2}}$ is a constant, say $c$, and these terms become

$$\frac{c}{2}\left(f_1''dx_1^2 - F_0''dx_0^2\right).$$

But $c$ is the sine of the inclination of $ab$ to the axis of $x$, and we may therefore so assume this axis as to render $c$ positive, and then we shall be sure of a minimum if $f_1''$ be positive while $F_0''$ is negative.

**73.** But it is unnecessary to pursue this investigation further. For it must now appear that the problem under consideration is one rather of the differential calculus than of the calculus of variations. For since we know from Prob. I. that the right line is the plain curve of minimum length between two points, whether they be situated upon given curves or not, we might have been certain beforehand that the solution could be no other curve than the right line, and that our problem could concern nothing but its position. Moreover, its position being determined, we need only compare the line with other right lines drawn to points on $ff$ and $gg$ consecutive to $a$ and $b$. For if we vary $ab$ so as to obtain a derived curve, $cd$, which is not exactly a right line, then, even if we show that $ab$ is shorter than $cd$, we could shorten $cd$ by making it a right line, its extremities remaining unchanged, and could not without a new comparison be certain that the new line $cd$ might not be shorter than $ab$.

The problem might then have been enunciated thus: *To find the position of the right time of minimum length which can be drawn between two given plane curves.*

**74.** Although problems of this sort might be altogether omitted here, there appears—at least so far as the terms of the first order are concerned—to be some advantage in solving

them by the calculus of variations instead of by the ordinary methods of *maxima* and *minima*. At all events, they are generally discussed by writers on this subject, and it is deemed necessary to render the reader familiar with the methods which they employ. We shall therefore subjoin a few more problems of the same kind, considering the terms of the first order only, since a discussion of those of the second would in general be unsatisfactory.

### Problem X.

**75.** *It is required to determine both the nature and position of the curve which will minimize the time of descent of a particle from one given curve to another, the particle starting from a fixed horizontal line, and being acted upon by gravity solely, all the curves lying in the same plane.*

Assume the fixed horizontal line as the axis of $x$, and let $x_0$ and $y_0$ be the co-ordinates of the point in which the required curve cuts the upper of the given curves, while $x_1$ and $y_1$ are the co-ordinates of the point in which it cuts the lower. Then, reasoning as we did in Prob. II., we see that we have to minimize the expression

$$U = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y}}\,dx = \int_{x_0}^{x_1} V dx.$$

Now it is clear that here, as in the preceding problem, the limits of integration will be also subject to variation. For suppose that after the required curve and its points of intersection with the given curves have been found, we assume points on the given curves consecutive to those just found, and then connect these new points by another curve. Then the abscissæ of these new points will be $x_0 + dx_0$ and $x_1 + dx_1$, $dx_0$ and $dx_1$ having either sign. It also appears, as before, that the total change which $U$ will undergo, both from a

change in the form of the curve and an alteration in the position of its extremities, can be found by first changing the limits of the integral in such a manner that the new limits may be the abscissæ of the new points, while the form of the curve remains unaltered, and then changing by the ordinary methods of variations the form of the curve taken throughout the new limits. By the change of limits only, $U$ becomes $U'$, where $U'$ is given by equation (1) of the preceding problem, because that equation will hold irrespectively of the form of $V$. Then if in $U'$ we change $y$ into $y + \delta y$, and $y'$ into $y' + \delta y'$, and subtract $U$, we shall have the exact value of $[\delta U]$, to which, however, we can only approximate. This approximation, so far as $U'$ is concerned, is effected as in equations (2) and (3) of the last problem, which also hold irrespectively of the form of $V$. If now we take the variation of $U'$ in the usual way, we shall have first the terms $\delta V_1 dx_1 - \delta V_0 dx_0$, which are evidently of the second order and must be rejected unless we are developing $[\delta U]$ to the second order, when they must be added to those involving $dx_1^2$ and $dx_0^2$. Next we obtain $\delta U$ or $\int_{x_0}^{x_1} \delta V dx$, where $\delta U$ is to be developed to the first or second order as required, and the terms of the first order transformed as in the case of fixed limits. Hence to the first order we shall have

$$[\delta U] = V_1 dx_1 - V_0 dx_0 + \int_{x_0}^{x_1} \delta V dx,$$

which equation would evidently hold irrespectively of the form of $V$.

But as in the present case $V$ contains $y$ and $y'$ only, if we put as usual $N$ for $\frac{dV}{dy}$ and $P$ for $\frac{dV}{dy'}$, and then develop $\delta V$ to the first order, and transform as usual, we shall have

$$[\delta U] = V_1 dx_1 - V_0 dx_0 + P_1 \delta y_1 - P_0 \delta y_0 + \int_{x_0}^{x_1} \left\{ N - \frac{dP}{dx} \right\} \delta y dx. (1)$$

**76.** Now it will be remembered that the relation expressed in either equation (10) of the preceding problem, not having been established upon any particular supposition, is true whatever be the equations of the limiting curves. In this case, therefore, if we assume $y=f(x)$ and $y=F(x)$ to be the equations of the two given curves, we can eliminate $\delta y_1$ and $\delta y_0$, the terms of the second order which result from the elimination being added to those already existing, or else being rejected if terms of the second order are not to be considered. When these terms are to be neglected, equations (10) are better written

$$\delta y_1 = (f'-y')_1\, dx_1, \qquad \delta y_0 = (F'-y')_0\, dx_0. \tag{2}$$

Performing this elimination, we have

$$[\delta U] = (V+Pf'-Py')_1\, dx_1 - (V+PF'-Py')_0\, dx_0$$
$$+\int_{x_0}^{x_1} \left\{ N - \frac{dP}{dx} \right\} \delta y\, dx = 0. \tag{3}$$

Now having equated the terms of the first order to zero, it will appear that, as the integral cannot be made to depend upon terms which relate solely to its limits without in some manner restricting the generality of the function $\delta y$, we can only satisfy the equation $[\delta U]=0$ by equating the integral and the terms at the limits separately to zero.

It will be seen from (3) that $[\delta U]$ and $\delta U$ differ only in the terms at the limits, the integrals being identical, and this would be the case if $V$ were any function whatever of $x, y, y', y''$, etc. Hence if we make the integral in (3) vanish, it must lead to the same general solution as though we had been discussing the problem of the brachistochrone between fixed points, and therefore the general solution must be a cycloid.

It is clear, also, that if $dx_0$ and $dx_1$ be entirely independent, as they are in this case, we can only make the terms at the limits

certainly vanish by equating severally to zero the coefficients of these quantities. Performing this operation, and substituting the values of $V$ and $P$, we obtain for the upper limit

$$\left\{ \frac{y'f' - y'^2}{\sqrt{y(1 + y'^2)}} + \frac{\sqrt{1 + y'^2}}{\sqrt{y}} \right\}_1 = 0,$$

whence by reduction we have

$$1 + y_1'f_1' = 0,$$

and in like manner, at the lower limit, we find

$$1 + y_0'F_0' = 0;$$

equations which show that the cycloid must cut each of the two given curves at right angles.

**77.** We see, then, from the preceding examples, that if we wish to determine the conditions which will maximize or minimize any single definite integral in which the limits also are to be subject to an indefinitely small change, we have merely to put the integral, if possible, under the form $U = \int_{x_0}^{x_1} V dx$, $V$ being some function of $x$, $y$, $y'$, $y''$, etc., and then, if the general solution be known in the case in which the limits are fixed, we need only consider the terms at the limits, as the general solution will in every case be the same, whether the limits be fixed or variable. Moreover, if we wish to consider the terms of the first order only, the terms at the limits in $[\delta U] = 0$ will be identical with those which occur in $\delta U = 0$, with the addition of the terms $V_1 dx_1 - V_0 dx_0$. Then if no restriction be imposed upon the quantities $dx_1$, $dx_0$, $\delta y_1$, $\delta y_0$, the coefficients of these quantities must be equated severally to zero. This would give us, in addition to the usual $2n$ conditions, $V_1 = 0$ and $V_0 = 0$, equations which, as we have already seen, could not in general be satisfied, as we would have

$2n+2$ equations and only $2n$ arbitrary constants. But when the extremities of the required curve are restricted to two given curves, we can eliminate $\delta y_1$ and $\delta y_0$ as already shown, and thus the number of ancillary equations is reduced once more to $2n$.

### Problem XI.

**78.** *It is required to determine the conditions which must hold at the limits, when in Prob. III. we also demand that the required curve shall have its extremities upon two given curves.*

Assume, as before, the differential equations of the curves to be $dy = f'dx, dy = F'dx$. Then, following the last article, we neglect all terms except those at the limits, since the general solution is known to be a cycloid. Here $V = \dfrac{-(1+y'^2)^2}{y''}$, and the terms at the limits, as will be seen from Art. 30, will, after adding $V_1 dx_1 - V_0 dx_0$, become

$$L = -\left(\frac{(1+y'^2)^2}{y''}\right)_1 dx_1 + \left(\frac{(1+y'^2)^2}{y''}\right)_0 dx_0$$

$$-\left(\frac{4y'(1+y'^2)}{y''} + \frac{d}{dx}\frac{(1+y'^2)^2}{y''^2}\right)_1 \delta y_1$$

$$+\left(\frac{4y'(1+y'^2)}{y''} + \frac{d}{dx}\frac{(1+y'^2)^2}{y''^2}\right)_0 \delta y_0$$

$$+\left(\frac{(1+y'^2)^2}{y''^2}\right)_1 \delta y_1' - \left(\frac{(1+y'^2)^2}{y''^2}\right)_0 \delta y_0' = 0. \qquad (1)$$

But from equation (11), Art. 30, we have

$$\frac{4y'(1+y'^2)}{y''} + \frac{d}{dx}\frac{(1+y'^2)^2}{y''^2} = -c.$$

Moreover, we shall assume that the cycloid has cusps at the points whose co-ordinates are suffixed, in consequence of which

$y''$ will become infinite, and the terms in (1) which are divided by $y''$ will vanish. Hence (1) becomes

$$L = -c(\delta y_1 - \delta y_0) = 0. \qquad (2)$$

But $\delta y_1 = (f' - y')_1 \, dx_1$, $\delta y_0 = (F' - y')_0 \, dx_0$, and substituting these values in (2), and equating severally to zero the coefficients of $dx_1$ and $dx_0$, we have

$$f_1' - y_1' = 0, \qquad F_0' - y_0' = 0.$$

But $y_1'$ and $y_0'$ are equal, because the tangents to the cycloids at its cusps are parallel, and therefore the quantities $y_1'$, $y_0'$, $f_1'$, $F_0'$ are equal. Hence we conclude that the chord joining the two cusps of the cycloid must be normal to each of the given curves.

### Problem XII.

**79.** *It is required that the generating curve in Prob. VII. shall have its extremities upon two given curves.*

Let the equations of the given curves be as in the preceding problem. Then $V = y\sqrt{1 + y'^2}$, and the terms at the limits become

$$L = (y\sqrt{1 + y'^2})_1 \, dx_1 - (y\sqrt{1 + y'^2})_0 \, dx_0$$

$$+ \left(\frac{yy'}{\sqrt{1 + y'^2}}\right)_1 \delta y_1 - \left(\frac{yy'}{\sqrt{1 + y'^2}}\right)_0 \delta y_0 = 0. \qquad (1)$$

Eliminating $\delta y_1$ and $\delta y_0$ as before, we have, after equating to zero the coefficients of $dx_1$ and $dx_0$,

$$\left\{y\sqrt{1 + y'^2} + \frac{yy'f'}{\sqrt{1 + y'^2}} - \frac{yy'^2}{\sqrt{1 + y'^2}}\right\}_1 = 0,$$

$$\left\{y\sqrt{1 + y'^2} + \frac{yy'F'}{\sqrt{1 + y'^2}} - \frac{yy'^2}{\sqrt{1 + y'^2}}\right\}_0 = 0.$$

Whence reducing, we have

$$1 + y_1'f_1' = 0, \qquad 1 + y_0'F_0' = 0,$$

which show that the catenary must cut its limiting curves at right angles.

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## SECTION IV.

### *CASE IN WHICH SOME OF THE LIMITING VALUES OF X, Y, Y', ETC., ENTER THE GENERAL FORM OF V.*

### Problem XIII.

**80.** *It is required to determine the nature and position of the curve down which a particle will descend in a minimum time from one given curve to another, all the curves being in the same vertical plane, and the motion of the particle beginning at the point of its departure from the upper curve.*

Assume the axis of $y$ vertically downward, and let $x_0$, $y_0$, $x_1$, $y_1$ be the respective co-ordinates of the initial and terminal points of the motion, and let the differential equations of the respective curves be $dy = F'dx$, and $dy = f'dx$. Now in this case the velocity of the particle at any point whose ordinate is $y$ will be $\sqrt{2g(y - y_0)}$, because the motion begins at the point whose ordinate is $y_0$. Therefore in this problem we must minimize the expression

$$U = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y - y_0}}\, dx = \int_{x_0}^{x_1} V dx. \tag{1}$$

Although it at once appears that the limits $x_0$ and $x_1$ will also be subject to change in this problem, we see that one of these limiting co-ordinates, $y_0$, enters likewise as a component part of $V$ throughout the integral, and this fact will require

some modification of our previous method of solution, because, since $y_0$ is a component of $V$, any change in the value of $y_0$ will produce a change in that of $V$ throughout the entire range of integration. Moreover, the co-ordinates at the lower limit must always satisfy the equation $y_0 = F(x_0)$, so that when we change $x_0$ into $x_0 + dx_0$, we necessarily change $y_0$ into $F(x_0 + dx_0)$. It happens that $V$ is not affected by any change in the values of the other limiting co-ordinates, as they do not occur in $V$; but if they did, the method of treatment would be similar to that which we are about to explain for $y_0$.

Now let $V'$ be what $V$ becomes when we change $y_0$ into $y_0 + dy_0$, and we shall have, from the change of limits only,

$$U' = \int_{x_0+dx_0}^{x_1+dx_1} V'dx. \tag{2}$$

If we next change $y$ into $y + \delta y$, and $y'$ into $y' + \delta y'$, and subtract $U$ or $\int_{x_0}^{x_1} Vdx$, the result will be the exact variation of $U$, to which we will now approximate as far as terms of the first order only. As before, to the first order, (2) becomes

$$U' = V_1'dx_1 - V_0'dx_0 + \int_{x_0}^{x_1} V'dx. \tag{3}$$

Now when we change $x_0$ into $x_0 + dx_0$, we to the first order change $y_0$ into $y_0 + F_0'dx_0$, and therefore $V'$ is what $V$ becomes when we change $y_0$ into $y_0 + F_0'dx_0$, $y$ and $y'$ in $V$ being regarded as constant, since they in no manner depend upon $y_0$; and this change in $V$ will evidently be $\frac{dV}{dy_0}F_0'dx_0$, where $F_0'dx_0$ has simply been put for $dy_0$. Hence to the first order,

$$V' = V + \frac{dV}{dy_0}F_0'dx_0. \tag{4}$$

Substituting this value in (3), rejecting again all terms of the

second order, and observing that $F_0'$ and $dx_0$ may be regarded as constants, we have

$$U' = V_1\,dx_1 - V_0\,dx_0 + F_0'dx_0\int_{x_0}^{x_1}\frac{dV}{dy_0}\,dx + \int_{x_0}^{x_1}V dx. \tag{5}$$

If now we vary $y$ and $y'$, we shall obtain the variation of $\int_{x_0}^{x_1}V dx$ or $U$ in the usual manner for fixed limits, while the variations of all the other terms must be neglected, being of an order higher than the first. Hence putting $N$ for $\frac{dV}{dy}$, $P$ for $\frac{dV}{dy'}$, we shall have, after the usual transformation of $\delta U$,

$$[\delta U] = V_1\,dx_1 - V_0\,dx_0 + P_1\delta y_1 - P_0\delta y_0 + F_0'dx_0\int_{x_0}^{x_1}\frac{dV}{dy_0}\,dx$$

$$+ \int_{x_0}^{x_1}\left\{N - \frac{dP}{dx}\right\}\delta y\,dx = 0. \tag{6}$$

But $\frac{dV}{dy_0} = -\frac{dV}{dy} = -N$, as will readily appear from the form of $V$ given in (1), so that we have

$$[\delta U] = V_1\,dx_1 - V_0\,dx_0 + P_1\delta y_1 - P_0\,\delta y_0 - F_0'\,dx_0\int_{x_0}^{x_1}N\,dx$$

$$+ \int_{x_0}^{x_1}M\,\delta y\,dx = 0, \tag{7}$$

where $M = N - \frac{dP}{dx}$. Now whether we can integrate the expression $\int_{x_0}^{x_1}N\,dx$ or not, we know that it is merely a function of the limiting co-ordinates and their differential coefficients, the form of the integral being dependent upon the nature of the general solution obtained by making the second member

of (7) zero, and it is not, therefore, in our power. Hence, by the same reasoning as before, we must have $\int_{x_0}^{x_1} M\,\delta y\,dx = 0$ and $M = 0$.

As we have merely assumed that the axis of $y$ shall be vertical, we may take that of $x$ so as to make $y_0$ zero, in which case the equation $M = 0$ will become identical with the same equation in Prob. II., Case 2, and the general solution will therefore be a cycloid—which solution will evidently also hold however we assume the axis of $x$, since by changing that, so long as that of $y$ is vertical, we change neither the form of the curve nor the values of any of the differential coefficients of $y$. The general solution then being a cycloid with a cusp on the upper curve, we must next, if possible, satisfy the equation

$$[\delta U] = V_1 dx_1 - V_0 dx_0 + P_1 \delta y_1 - P_0 \delta y_0 - F_0' dx_0 \int_{x_0}^{x_1} N\,dx = 0. \quad (8)$$

Now in this case the equation $M = 0$ gives $N = \frac{dP}{dx}$, where the differential is total. Hence

$$- F_0' dx_0 \int_{x_0}^{x_1} N\,dx = - F_0' dx_0 (P_1 - P_0). \quad (9)$$

Now substitute this value in (8), and next eliminate $\delta y_1$ and $\delta y_0$ by equations (2), Art. 76. Then equating to zero severally the coefficients of $dx_1$ and $dx_0$, we have

$$V_1 + P_1 f_1' - P_1 y_1' = 0, \quad (10)$$

$$V_0 + P_1 F_0' - P_0 y_0' = 0. \quad (11)$$

Since the general solution is a cycloid, we have, from the equation $y(1 + y'^2) = a$ of Art. 25, by putting $y - y_0$ for $y$,

$$\sqrt{(y - y_0)(1 + y'^2)} = \sqrt{a} = \sqrt{2r}.$$

Substituting this value in (10) and (11) after having restored the values of $V$ and $P$, they become after reduction

$$1 + y_1'f_1' = 0, \qquad 1 + y_1'F_2' = 0,$$

equations which show that the cycloid cuts the lower curve at right angles, while, since $f_1' = F_2'$, the tangents of the two given curves at the initial and terminal points of the motion must be parallel.

**81.** We have seen that when a particle starts from a state of rest, the cycloid must have a cusp at that point. But if it is to start with a given initial velocity in the direction of the tangent, which velocity could always have been produced by falling from some height $h$, $V$ in Prob. II., Case 2, would become

$$\frac{\sqrt{1+y'^2}}{\sqrt{y+h}}.$$

If, as usual, we obtain the differential equation

$$N - \frac{dP}{dx} = M = 0,$$

we can evidently, while keeping $y$ vertical, remove the axis of $x$ to the height $h$ above the initial point, without affecting the form of the curve given by the equation $M = 0$. But making this change, $y + h$ will become $y$, and $M$ will become identical with $M$ in Prob. II., Case 2, thus giving us a cycloid with its cusps upon the new axis of $x$. That is, when the particle starts with a given tangential velocity, the curve of quickest descent, or the brachistochrone, will still be a cycloid, but having its cusps upon the horizontal, from which the particle must have fallen in order to acquire the given initial velocity upon reaching the starting-point.

In like manner, in the last problem, if we require the par-

ticle to start from the upper curve with a fixed tangential velocity, due to some height $h$, $V$ will merely become

$$\frac{\sqrt{1+y'^2}}{\sqrt{h+y-y_0}},$$

and no change will be effected in the results of the last article, except that the cycloid will no longer have a cusp upon the upper curve, but its cusps will then be upon the horizontal whose distance above the upper intersection is $h$.

**82.** As examples of the kind discussed in the preceding problem are not numerous, we shall, as a means of more fully developing the method therein explained, now examine the terms of the second order.

For greater simplicity, change the independent variable, assuming the axis of $x$ vertically downward; and for greater generality, suppose the particle to start from the upper curve with an initial tangential velocity due to the height $h$. Also let the equations of the curves be $y = F(x) = F$ for the upper, and $y = f(x) = f$ for the lower, while $x_0, y_0, x_1, y_1$, are the coordinates of the initial and terminal points of the motion. Now we shall have

$$V = \frac{\sqrt{1+y'^2}}{\sqrt{h+x-x_0}}.$$

Let $V'$ be at once what $V$ becomes when $y'$ is changed into $y'+\delta y'$, and $x_0$ into $x_0+dx_0$. Then we have

$$[\delta U] = \int_{x_0+dx_0}^{x_1+dx_1} V'dx - \int_{x_0}^{x_1} Vdx, \tag{1}$$

which is exact; and we will now approximate to the second order. We have

$$V' = V + \frac{dV}{dy'}\delta y' + \frac{dV}{dx_0}dx_0$$

$$+\frac{1}{2}\frac{d^2V}{dy'^2}\delta y'^2 + \frac{d^2V}{dy'dx_0}\delta y' dx_0 + \frac{1}{2}\frac{d^2V}{dx_0^2}dx_0^2. \qquad (2)$$

For brevity, let $A$ denote all the terms of the first order except $V$, $B$ those of the second, and $C$ their sum. Then (1) becomes

$$[\delta U] = \int_{x_0+dx_0}^{x_1+dx_1} V dx - \int_{x_0}^{x_1} V dx + \int_{x_0+dx_0}^{x_1+dx_1} C dx. \qquad (3)$$

But, as formerly, the first integral in (3) gives

$$V_1 dx_1 - V_0 dx_0 + \frac{1}{2}\left(\frac{dV}{dx}\right)_1 dx_1^2 - \frac{1}{2}\left(\frac{dV}{dx}\right)_0 dx_0^2 + \int_{x_0}^{x_1} V dx. \qquad (4)$$

Moreover, neglecting terms of an order higher than the second, the last integral in (3) becomes

$$\int_{x_0+dx_0}^{x_1+dx_1} A\, dx + \int_{x_0}^{x_1} B\, dx. \qquad (5)$$

Also to the second order

$$\int_{x_0+dx_0}^{x_1+dx_1} A\, dx = A_1 dx_1 + A_0 dx_0 + \int_{x_0}^{x_1} A\, dx. \qquad (6)$$

Hence, finally, we have

$$[\delta U] = V_1 dx_1 - V_0 dx_0 + \frac{1}{2}\left(\frac{dV}{dx}\right)_1 dx_1^2 - \frac{1}{2}\left(\frac{dV}{dx}\right)_0 dx_0^2$$

$$+ A_1 dx_1 - A_0 dx_0 + \int_{x_0}^{x_1} C dx. \qquad (7)$$

Restoring the value of $A$, transforming by integration, as

usual, the term $\int_{x_0}^{x_1} \frac{dV}{dy'} \delta y' dx$, and then eliminating $\delta y_1$ and $\delta y_0$ by equations (10), Art. 69, we have

$$[\delta U] = V_1 dx_1 - V_0 dx_0$$

$$+\left(\frac{dV}{dy'}\right)_1 (f' - y')_1 dx_1 - \left(\frac{dV}{dy'}\right)_0 (F' - y')_0 dx_0 + dx_0 \int_{x_0}^{x_1} \frac{dV}{dx_0} dx$$

$$- \int_{x_0}^{x_1} \frac{d}{dx} \frac{dV}{dy'} \delta y \, dx + \frac{1}{2} \left(\frac{dV}{dx}\right)_1 dx_1^2 - \frac{1}{2} \left(\frac{dV}{dx}\right)_0 dx_0^2$$

$$+ \frac{1}{2} \left(\frac{dV}{dy'}\right)_1 (f'' - y'')_1 dx_1^2 - \frac{1}{2} \left(\frac{dV}{dy'}\right)_0 (F'' - y'')_0 dx_0^2 - \left(\frac{dV}{dy'}\right)_1 \delta y_1' dx_1$$

$$+ \left(\frac{dV}{dy'}\right)_0 \delta y_0' dx_0 + \left(\frac{dV}{dy'} \delta y' + \frac{dV}{dx_0} dx_0\right)_1 dx_1 - \left(\frac{dV}{dy'} \delta y' + \frac{dV}{dx_0} dx_0\right)_0 dx_0$$

$$+ \int_{x_0}^{x_1} B \, dx. \tag{8}$$

Making the terms of the first order vanish, we shall, as before, obtain the cycloid as the general solution, and it will be subject to the conditions already explained. Then $[\delta U]$ will consist of the terms of the second order only, which must become positive if the solution give a true minimum. As the terms in $\delta y'$ cancel, we shall have

$$[\delta U] = \frac{1}{2} \left(\frac{dV}{dy'}\right)_1 (f'' - y'')_1 dx_1^2 - \frac{1}{2} \left(\frac{dV}{dy'}\right)_0 (F'' - y'')_0 dx_0^2$$

$$+ \frac{1}{2} \left(\frac{dV}{dx}\right)_1 dx_1^2 - \frac{1}{2} \left(\frac{dV}{dx}\right)_0 dx_0^2 + \left(\frac{dV}{dx_0}\right)_1 dx_0 \, dx_1 - \left(\frac{dV}{dx_0}\right)_0 dx_0^2$$

$$+ \int_{x_0}^{x_1} B \, dx. \tag{9}$$

Now we cannot render it evident that this value of $[\delta U]$ is necessarily positive, nor will any of our subsequent investigations afford us the required assistance, there being no known method applicable. Therefore, although the great Legendre erroneously supposed that we were sure of a minimum, we cannot in fact be certain of its existence in every case. (See Todhunter's History of Variations, Arts. 202, 300.)

**83.** When $V$ contains several of the quantities $x_0$, $y_0$, $y_0'$, $x_1$, $y_1$, $y_1'$, etc., the expression for $[\delta U]$ becomes somewhat complicated. But as we know that to the first order the change which any function undergoes from an indefinitely small alteration in any of its components may be found by considering each change separately and then taking their sum, we may, as Prof. Jellett has suggested, use this method with advantage here, as we shall not require the terms of the second order.

Suppose, then, that we have to maximize or minimize the expression $U=\int_{x_0}^{x_1} Vdx$, where $V$ is a function of $x, y, y'$, and also some of the limiting values of these quantities, $x_0$ and $x_1$ being subject to change into $x_0+dx_0$ and $x_1+dx_1$. From the change in $x_0$ alone, supposing the other quantities could remain unaltered, $U$ will be increased by $V_1 dx_1 - V_0 dx_0$. From varying $y, y'$, etc., $V$ would be increased by $\frac{dV}{dy}\delta y+\frac{dV}{dy'}\delta y'+$ etc., or $\delta V$, and $U$ would therefore be increased by $\int_{x_0}^{x_1}\delta V dx$. Lastly, by the alteration in the limiting quantities which enter it, $V$ would, throughout the entire range of integration, be increased by $\frac{dV}{dx_1}dx_1+\frac{dV}{dy_1}\delta y_1+\frac{dV}{dy_1'}\delta y_1'+$ etc., and the same for the lower limit. Calling this change $\delta' V$, $U$ is increased by $\int_{x_0}^{x_1}\delta' V dx$. Adding these results, we have

$$[\delta U] = V_1 dx_1 - V_0 dx_0 + \int_{x_0}^{x_1} \delta' V dx + \int_{x_0}^{x_1} \delta V dx = 0. \quad (1)$$

Now the last integral in (1), being transformed as usual, will give us, besides certain additional terms at the limits, a differential equation $M = 0$, and this equation will be the same in form as though $V$ had not contained any of the limiting components. Hence the general solution will be the same as though $V$ had not contained these quantities, and the limits also had been fixed. Then, by using this general solution, we must if possible, by definite integration, express the remaining integral in terms of suffixed quantities, our power to complete the solution being dependent upon our ability to remove this integral sign. After this has been done, we discuss the resulting limiting equations as we would in any other case.

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## SECTION V.

*CASE IN WHICH U IS A MIXED EXPRESSION; THAT IS, CONTAINS AN INTEGRAL, TOGETHER WITH TERMS FREE FROM THE INTEGRAL SIGN.*

### Problem XIV.

**84.** *It is required to maximize or minimize the expression*

$$U = \int_{x_0}^{x_1} y^n \frac{y''}{y'} dx = \int_{x_0}^{x_1} V dx.$$

Here $V$ is a function of $y$, $y'$, $y''$, whence, by formula (A), Art. 56,

$$V = c + Py' + Qy'' - y' \frac{dQ}{dx}, \quad (1)$$

and

$$P = -y^n \frac{y''}{y'^2}, \qquad Q = \frac{y^n}{y'}, \qquad \frac{dQ}{dx} = ny^{n-1} - y^n \frac{y''}{y'^2}.$$

Hence (1) gives

$$ny^{n-1}y' = c, \tag{2}$$

and, by integration,

$$y^n = cx + d. \tag{3}$$

Now the terms at the upper limit are

$$\left(P - \frac{dQ}{dx}\right)_1 \delta y_1 + Q_1 \delta y_1',$$

and similar terms at the lower limit, so that unless some restriction be imposed upon the independence of $\delta y_1$, $\delta y_0$, $\delta y_1'$, and $\delta y_0'$, there will be four limiting equations to satisfy, while the general solution contains but two arbitrary constants, and this will in general be impossible.

But the above example, containing the first power only of $y''$, the highest differential coefficient in $V$, is, as will be remembered, a case of Exception 2, Art. 51. It will also be remembered that it was shown by Euler's method, equation (8), Art. 52, that all such integrals can be reduced to a lower order, the expression taking the form $W_1 - W_0 + \int_{x_0}^{x_1} V' dx$, a class of problems not yet considered. In the present case, recollecting that $y''dx = dy'$, $y'dx = dy$, we easily obtain, by parts,

$$\int_{x_0}^{x_1} y^n \frac{y''}{y'} dx = y_1^n l y_1' - y_0^n l y_0' - \int_{x_0}^{x_1} n y^{n-1} y' l y' dx$$

$$= W_1 - W_0 + \int_{x_0}^{x_1} V' dx. \tag{4}$$

Now if we vary $y_1$, $y_1'$, we shall increase $W_1$ by

$$\frac{dW_1}{dy_1}\delta y_1 + \frac{dW_1}{dy_1'}\delta y_1', \qquad \text{or } (ny^{n-1}\, l y' \delta y)_1 + \left(\frac{y^n}{y'}\delta y'\right)_1,$$

and we can change $W_1$ in no other way. A similar equation of course holds for $W_0$. But these terms, relating to the limits only, can have nothing to do with the form of any general solution, which must, therefore, depend solely upon the form of $V'$.

Now $V'$ is a function of $y$ and $y'$ only, and

$$P = -ny^{n-1}ly' - ny^{n-1}.$$

Hence by formula (C), Art. 56, we have, as before,

$$ny^{n-1}y' = c, \qquad y^n = cx + d.$$

Now the terms at the limits resulting from the variation of $V'$ are $P_1\delta y_1 - P_0\delta y_0$, which must be added to those obtained by varying $W_1$ and $W_0$. Performing this operation, these terms become

$$-(ny^{n-1})_1\delta y_1 + (ny^{n-1})_0\delta y_0 + \left(\frac{y^n}{y'}\right)_1\delta y_1' - \left(\frac{y^n}{y'}\right)_0\delta y_0'.$$

But these terms are the same as those which we obtained by discussing the problem as originally given; and as the general solution is also the same, the difficulty which formerly occurred is not removed.

**85.** We may, however, from this example see how to proceed in more important cases of mixed integrals which will hereafter occur. Thus, suppose we have to maximize or minimize the expression

$$U = W_1 - W_0 + \int_{x_0}^{x_1} V dx,$$

where $W_1$ and $W_0$ are any functions of $x_1$, $y_1$, $y_1'$, etc., and $x_0$, $y_0$, $y_0'$ . . . . $y^{(m)}$, and $V$ is any function of $x$, $y$, $y'$ . . . . $y^{(n)}$. while the limiting values of $x$ are also variable. As before,

if we change $x_1$ into $x_1 + dx_1$ and vary $y_1$, $y_1'$, etc., $W_1$ will receive the increment

$$\frac{dW_1}{dx_1}dx_1 + \frac{dW_1}{dy_1}\delta y_1 + \frac{dW_1}{dy_1'}\delta y_1' + \text{etc.},$$

and $W_0$ will be increased in a similar manner by changing $x_0$ into $x_0 + dx_0$ and varying $y_0$, $y_0'$, etc. These terms, being all suffixed, cannot control the general solution, which must be obtained by varying $V$ in the usual manner, transforming the variation as previously explained, and solving the differential equation $M = 0$ which will be obtained. Then we have as the terms at the limits those derived from the transformation of $\delta V$, together with those derived from varying $W_1$ and $W_0$. Now if the limits be fixed we shall generally, in order that the number of limiting equations may not exceed that of the constants in the general solution, require that $m$ shall not exceed $n - 1$, the difficulty in the last problem arising from the fact that $m$ is equal to $n$. But if the limits be not fixed, we shall also, as before, require usually some restriction upon the extremities of the curve given by the general solution.

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## SECTION VI.

### *RELATIVE MAXIMA AND MINIMA.*

### Problem XV.

**86.** *It is required to find among all plane curves of a given length which can be drawn between two fixed points, that which, together with the ordinates of its extremities and the axis of x, shall contain a maximum area.*

Whatever be the nature of the required curve, we know that its length is $\int_{x_0}^{x_1} \sqrt{1 + y'^2}\,dx$; and since it is to be com-

pared with curves of the same length only, its derived curves must not differ from it in length, and we must therefore have $\int_{x_0}^{x_1}\delta\sqrt{1+y'^2}dx = 0$. But the enclosed area is $\int_{x_0}^{x_1}ydx$; and since this is to be a maximum for all changes in the form of the curve which permit its length to remain unaltered, we must have also to the first order $\int_{x_0}^{x_1}\delta y\,dx = 0$.

Now in the problems hitherto considered no restriction has been imposed upon the variations of $y$, $y'$, etc., except that they must always be infinitesimal, and the curves given by the general solution have therefore been compared with all others which can be derived from them by such variations. The results, therefore, being subject to no restriction so far as the variations are concerned, are termed *absolute maxima and minima*, observing that the terms maxima and minima are used in their technical sense only, and not in that of greatest or least. But in the present problem we are to compare the required curve with such only as can be derived by infinitesimal variations of $y'$ without any increase in its length, and the area is to be a maximum with respect to such variations only. That is, if we vary the required curve so as to increase its length, the area need no longer be a maximum. Examples of this nature, therefore, are termed problems of *relative maxima and minima*, and also *isoperimetrical problems*, and constitute the most numerous and important class of questions discussed in the calculus of variations.

**87.** Resuming the equations of the last article, and treating the first as usual, recollecting that $\delta y_1$ and $\delta y_0$ are zero, we have

$$\int_{x_0}^{x_1}\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\,\delta y\,dx = 0, \tag{1}$$

$$\int_{x_0}^{x_1}\delta y\,dx = 0, \tag{2}$$

which signify merely that any values of $\delta y$ which will satisfy (1) must also satisfy (2), it being supposed that the derived curve has been obtained. But although we are permitted to pass from the required curve to such derived curves only as do not differ from it in length, the number of such curves may nevertheless be infinite, so that we cannot express in an explicit form the nature of the restriction which has been imposed upon $\delta y$, or rather upon $\delta y'$, although we know that such variations could be given to $y'$ as would not satisfy equation (1), and might or might not satisfy (2). This restriction prevents us from employing our former reasoning, which would here give the equations $\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0$, the differential equation of the right line, as appears from Prob. I., and the impossible equation $1 = 0$. Now put $Z$ for $\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}$. Then if $Z$ can be a constant, it is evident that any values of $\delta y$ or $\delta y'$ which will satisfy one of the equations at the beginning of the last article will satisfy the other also; and we will now show that this is the only condition which will insure that $\delta y$ cannot be so taken as to satisfy one equation and not the other.

**88.** Let $x_2$, $x_3$, $x_4$, $x_5$ be four particular values of $x$ chosen as hereafter explained, and let $s$ denote the value of the integral $\int \delta y\, dx$ when the limits are $x_2$ and $x_3$, and $t$ its value when the limits are $x_4$ and $x_5$. Then supposing the required curve to be obtained, let us make $\delta y$ zero, except from $x_2$ to $x_3$, and from $x_4$ to $x_5$; that is, leave the required curve unvaried in form except between these limits. Also let us give to $\delta y$ an invariable sign from $x_2$ to $x_3$, and an invariable but contrary sign from $x_4$ to $x_5$. Then we shall have

$$\int_{x_0}^{x_1} \delta y\, dx = s + t. \tag{3}$$

Now although neither $s$ nor $t$ separately vanishes, we can so take $\delta y$ that their sum shall vanish, and thus (1) be satisfied. Next let $q$ denote the value of the integral $\int Z\delta y\,dx$ when the limits are $x_2$ and $x_3$, and $r$ its value when the limits are $x_4$ and $x_5$. Now the four values of $x$ may also be so taken that $Z$ will be of invariable sign from $x_2$ to $x_3$, and also from $x_4$ to $x_5$, it being of no importance whether the signs be the same or not for these two intervals. We can now, with the values of $\delta y$ formerly chosen, secure that, unless $Z$ be a constant, $q$ and $r$ shall be numerically unequal, and consequently that their sum shall not vanish. But, as before,

$$\int_{x_0}^{x_1} Z\,\delta y\,dx = q + r, \tag{4}$$

and hence, if $Z$ be variable, we can, without violating the restriction which has been put upon $\delta y$, give it such values as will satisfy equation (2) but not (1), which is contrary to the conditions of the question.

**89.** Now since $Z$ is a constant, let it equal $\frac{1}{a}$. Then $aZ = 1$; and restoring the value of $Z$, we may write

$$1 - a\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0, \tag{5}$$

an equation which involves the coefficients of $\delta y\,dx$ in both (1) and (2), and is necessarily true, being equivalent to $1 - 1 = 0$. But it will be seen that this differential equation, which combines both conditions of the question, would also have been obtained if we had at first required to maximize or minimize the expression

$$U = \int_{x_0}^{x_1}\left(y + a\sqrt{1+y'^2}\right)dx,$$

the extreme co-ordinates being fixed, and $\delta y$ or $\delta y'$ being subject to no restriction. Moreover, we shall presently show that all problems of this sort can be treated in a similar manner.

Now integrating (5), we have

$$x - \frac{ay'}{\sqrt{1+y'^2}} = c, \qquad \frac{ay'}{\sqrt{1+y'^2}} = -(c-x); \tag{6}$$

and solving for $y'$, we have

$$y' = \frac{c-x}{\sqrt{a^2-(c-x)^2}}. \tag{7}$$

Whence, by integration, we obtain

$$y + d = \sqrt{a^2-(c-x)^2}, \tag{8}$$

which shows, if we employ, as we have, the positive sign, that the required curve must be a circular arc, in which $a$ must be numerically equal to the radius $r$.

**90.** Suppose now, as just suggested, we attempt to maximize or minimize absolutely the expression

$$U = \int_{x_0}^{x_1} \left(y + a\sqrt{1+y'^2}\right) dx = \int_{x_0}^{x_1} V dx.$$

Here $V$ is a function of $y$ and $y'$, and $P = \dfrac{ay'}{\sqrt{1+y'^2}}$, so that by formula (C), Art. 56, we have

$$y + a\sqrt{1+y'^2} = c + \frac{ay'^2}{\sqrt{1+y'^2}}.$$

Whence

$$y + \frac{a}{\sqrt{1+y'^2}} = c, \qquad 1 + y'^2 = \frac{a^2}{(c-y)^2},$$

which must be solved thus:

$$dx = \frac{(c - y)\, dy}{\sqrt{a^2 - (c - y)^2}},$$

where we still use the positive sign. Integrating this equation, we have

$$x + d = \sqrt{a^2 - (c - y)^2}, \tag{9}$$

which evidently has the same interpretation as before, except that $c$ and $d$ need not be identical with $c$ and $d$ of the last article.

**91.** It will be seen that besides the two constants which arise from the integration of (5), which we may call $M = 0$, we have also a third constant, $a$. But now we also have, besides the two ordinary conditions given by assigning the values of $y_1$ and $y_0$, a third condition, that the length of the circular arc shall have an assigned value; and these conditions are sufficient for the determination of the three constants.

Consider first the constant $a$. We know that the length of the chord of the given arc is $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$, and is, therefore, determined as soon as the limiting co-ordinates are given; and since the length of the arc is assigned, if we find an expression for the length of any arc in terms of its chord and radius, and then substitute in that expression the known values of the chord and arc in question, we can, by solving for $a$, determine its value definitely. This expression is

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = 4r^2 \sin^2 \frac{s}{2r}, \tag{10}$$

where $s$ is the length of the given arc, and $a$ is numerically equal to $r$, its sign being reserved for future discussion.

After the determination of $a$, the other constants are readily found. For, from the general equation of the circle, we have

$$y_1 + d = \pm \sqrt{a^2 - (c - x)^2},$$

and a similar equation for the lower limit; and from these two equations, when the sign of their second members has been agreed upon, $c$ and $d$ can evidently be expressed in terms of $a$ and the given limiting co-ordinates.

**92.** We will now, before proceeding further, consider the general mode of treating problems of relative maxima and minima.

Suppose, then, we require that $\int_{x_0}^{x_1} v\,dx$ shall be a maximum or minimum, $v$ being any function of $x, y, y' \ldots y^{(n)}$, while at the same time $\int_{x_0}^{x_1} v'dx$ is to remain always constant, $v'$ being any other function of $x, y, y' \ldots y^{(m)}$. Then because $\int_{x_0}^{x_1} v\,dx$ is to be a maximum or minimum, we shall have to the first order

$$\int_{x_0}^{x_1} \delta v\,dx = 0; \tag{1}$$

and because $\int_{x_0}^{x_1} v'dx$ is to be always constant, we must have absolutely

$$\int_{x_0}^{x_1} \delta v'dx = 0. \tag{2}$$

Now suppose the variations of these integrals to be found, and transformed by integration in the usual manner. Then if we make $\delta y_1$, $\delta y_0$, $\delta y_1'$, etc., zero, we shall obtain, from (1) and (2) respectively, results of the form

$$\int_{x_0}^{x_1} V\delta y\,dx = 0, \tag{3}$$

$$\int_{x_0}^{x_1} V'\delta y\,dx = 0. \tag{4}$$

But $\delta y$ being restricted, as hitherto explained, we cannot say that $V$ and $V'$ must separately vanish, but equations (3) and (4) will certainly be satisfied simultaneously if we can be sure that $V'$ is always equal to $V$ multiplied by some constant; that is, that $\frac{V'}{V}$ is a constant; and we will now show that no other condition will satisfy these equations simultaneously.

**93.** Supposing the required curve to have been obtained, choose, as before, four values of $x$ such that neither $V$ nor $V'$ shall change its sign while $x$ lies between $x_2$ and $x_3$ or between $x_4$ and $x_5$. Now, as previously, vary the form of the curve between these two intervals only, and make the sign of $\delta y$ invariable for each interval separately, giving to it the same or contrary sign for these two intervals, according as that of $V$ is contrary or the same. Then, although $\int V\delta y\, dx$ does not vanish when taken throughout either interval separately, we can so vary $y$ as to make the integral taken throughout the second equal to the same integral taken throughout the first, but with a contrary sign. But we have

$$\int_{x_0}^{x_1} V\,\delta y\, dx = \int_{x_2}^{x_3} V\,\delta y\, dx + \int_{x_4}^{x_5} V\,\delta y\, dx, \tag{5}$$

$\delta y$ being zero for the rest of the curve. Therefore (2) would in this case be satisfied. Now put $f$ for $\frac{I''}{I'}$, then (4) will become

$$\int_{x_0}^{x_1} f\, V\,\delta y\, dx = 0, \tag{6}$$

$\delta y$ being supposed taken as before. But unless $f$ be a constant, we can certainly select the four values of $x$ so that the two integrals in the second member of (6) shall be numerically unequal, in which case their sum would not vanish and

(6), or rather (4), would not be satisfied. Hence $f$ must be a constant in order to the existence of a relative maximum or minimum, since then any values of $\delta y$ which will satisfy (3) will also satisfy (4), while otherwise it would be possible, even from among the restricted values of $\delta y$, to select such as would satisfy one of these equations and not the other.

The preceding demonstration is due to Bertrand (see Todhunter's History of Variations, Art. 312, and also the seventh volume of Liouville's Mathematical Journal, 1842), and the author most heartily agrees with Bertrand in regarding the ordinary method of treating this subject as insufficient.

Now write

$$a = -\frac{1}{f} = -\frac{V}{V'},$$

then

$$V + aV' = V - V = 0.$$

But this equation, which involves $V$ and $V'$, and, being true under all circumstances, is evidently sufficient for the solution of the problem, would have been obtained if we had been seeking to render $U$ an absolute maximum or minimum, where $U = \int_{x_0}^{x_1} (v + av')\,dx$, and thus we are enabled to substitute for the given problem a problem of absolute maximum or minimum, the general solution of which will be identical with that which we require.

This method is due to the illustrious Euler, who first reduced the treatment of this class of problems to a simple yet comprehensive rule. (See Jellett, Introduction, page xvii.)

It is evidently immaterial which of the quantities $v$ and $v'$ we select to be multiplied by a constant. For if we have $V = aV' = 0$, then $V' + bV = 0$, where $b = \frac{1}{a}$. Moreover, we may also give the constant multiplier any form which may be convenient, as $-a$, $2a$, etc., its value being ascertained subsequently.

**94.** Resuming the consideration of Prob. XV., let us now examine the terms of the second order. Here a difficulty presents itself in the outset which must be surmounted before we can proceed. We find that the variation of the area is simply $\int_{x_0}^{x_1} \delta y\, dx$, there being no additional terms of the second order; so that if we equate this variation to zero, it would seem that the area could undergo no change whatever when the curve is varied, and that consequently we could have neither a maximum nor a minimum. But the supposition that the terms of the first order must vanish is only necessary when there are terms of a higher order, it being sufficient, in a case like the present, to suppose that they are zero so far as the terms of the first order are concerned; that is, they need not be zero as regards $\delta y^2$. Nevertheless, as we cannot determine the nature of these terms of the second order, should any exist, we shall be compelled to change our method of investigation.

Suppose, then, that we had required the curve of minimum length which, together with its extreme ordinates and the axis of $x$, shall enclose a given area. Here the general solution will evidently be the same as formerly. For proceeding as in the first three articles of this section, we shall obtain equations identical with (1) and (2); and moreover, by the last article, we see that by Euler's method we are now merely to maximize or minimize the expression

$$U = \int_{x_0}^{x_1} \left( \sqrt{1 + y'^2} + by \right) dx,$$

where $b = \frac{1}{a}$. But the enclosed area, instead of being a maximum, is now to be constant, so that $\int_{x_0}^{x_1} \delta y\, dx$ is absolutely zero; while the length of the required curve, instead of being constant, is now to become a minimum.

It should here also be noticed that while the length of the required curve was to be constant, equation (1), Art. 87, can be true to the first order only. For since the variation of the length contains terms of an order higher than the first, and the entire series is to vanish absolutely, it is clear that the term of the first order must equal the sum of the others, taken with a contrary sign.

As the area gives us no term of the second order, we have only that obtained from the variation of the required curve, which is

$$\int_{x_0}^{x_1} \frac{1}{2\sqrt{(1+y'^2)^3}}\,\delta y'^2 dx\,; \qquad (1)$$

and if we regard $\sqrt{1+y'^2}$ as positive, the length of the curve is evidently a minimum. It must, however, be remembered that $\delta y$ and $\delta y'$ are restricted to such values only as will satisfy the equation $\int_{x_0}^{x_1} \delta y\, dx = 0$. But since (1) is positive for all real values of $\delta y'$, we only require that the term of the first order in the variation of the length of the curve shall completely vanish to insure a minimum; and since $\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}$ is a constant, this condition is secured when we make $\int_{x_0}^{x_1} \delta y\, dx$ vanish absolutely.

It will be seen that equation (1) would have been obtained had we found, according to Euler's method, the terms of the second order in

$$U = \int_{x_0}^{x_1} \left(\sqrt{1+y'^2} + by\right) dx,$$

$b$ being $\frac{1}{a}$. But, as before, the variations are not entirely unrestricted, since they can have such values only as will make $\int_{x_0}^{x_1} \delta y\, dx$ vanish absolutely.

**95.** Now let us, according to Euler's method, consider the problem as originally given. Then we shall have

$$U = \int_{x_0}^{x_1} \left(y + a\sqrt{1 + y'^2}\right) dx. \tag{2}$$

Here

$$v = y, \qquad v' = \sqrt{1 + y'^2}, \qquad V = 1, \qquad V' = -\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}}.$$

Hence

$$f = -\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = -\frac{y''}{\sqrt{(1 + y'^2)^3}} = \mp\frac{1}{r}, \tag{3}$$

where the last member has the negative or positive sign according as the circular arc is convex or concave to the axis of $x$. Therefore $a = -\dfrac{1}{f} = \pm r$, the positive or negative sign being used according as the circular arc is convex or concave to the axis of $x$. Making the terms of the first order vanish, (2) will give

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \frac{a}{\sqrt{(1 + y'^2)^3}}\, \delta y'^2\, dx = \pm\frac{r}{2}\int_{x_0}^{x_1} \frac{1}{\sqrt{(1 + y'^2)^3}}\delta y'^2 dx. \tag{4}$$

Hence if the arc be concave to the axis of $x$, the area is a maximum; but if convex, the area becomes a minimum; and these results are evidently as they should be.

It must, however, be remembered that we have not as yet shown that the use of Euler's method, as far as the terms of the second order, must in this latter case give necessarily a trustworthy result, but merely that this result is one which is known to be true.

**96.** We may now extend the discussion of this problem, and also that of relative maxima and minima generally, to the case

in which the limiting values of $x, y, y'$, etc., are also subject to change. We have already seen that if we seek to maximize or minimize an integral of the form $U = \int_{x_0}^{x_1} V dx$, the general solution will be the same whether we suppose the limiting values of $x, y, y'$, etc., to be fixed or not. Hence when $V$ becomes, as it will in the use of Euler's method, $v + av'$, the general solution, obtained under the supposition that the limits are variable, will be identical in form with that obtained by supposing those limits to be fixed. Now suppose we add to Prob. XV. the condition that the required curve shall always have its extremities upon two given curves; and suppose that the two required points had been found and connected by the required curve. Then, unless this curve were a circular arc, we could evidently, from our preceding discussion, vary it so as to increase the required area without changing the extremities of the curve. The general solution must therefore, as formerly, be a circular arc, the only question being to determine the position of its extremities.

The reader will be able at once to apply a similar mode of reasoning to any problem of relative maxima or minima which may present itself; and therefore, without taking space to generalize the demonstration, we shall assume that the general solution of these problems is, like those of absolute maxima and minima, the same in form whether the limiting values of $x, y, y'$, etc., be fixed or variable. Hence, from what has been said, we see that Euler's method may be employed whether the limits of integration be fixed or variable, the problem being treated in all respects like one of absolute maxima or minima.

**97.** Assume, then, in order to discuss the limits,

$$U = \int_{x_0}^{x_1} \left( y \pm r\sqrt{1 + y'^2} \right) dx. \tag{1}$$

If we suppose first that $x_1$ and $x_0$ are fixed, while $y_1$ and $y_0$ are variable—that is, that the arc has its extremities upon the two right lines whose equations are $x = x_0$ and $x = x_1$—the terms at the limits evidently become

$$\pm r\left(\frac{y'}{\sqrt{1+y'^2}}\right)_1 = 0, \qquad \pm r\left(\frac{y'}{\sqrt{1+y'^2}}\right)_0 = 0,$$

which equations signify that the tangent to the arc at each limit must be parallel to $x$, which is clearly impossible. But if one of the limiting values of $y$ be fixed, the tangent at the other limit can be drawn as described, and it must be so drawn.

Now suppose that the limiting values of $x$ are to be variable also. Then the terms at the limits will evidently give the equation

$$\left(y \pm r\sqrt{1+y'^2}\right)_1 dx_1 \pm r\left(\frac{y'}{\sqrt{1+y'^2}}\right)_1 \delta y_1 = 0, \tag{2}$$

with a similar equation at the lower limit. Let the extremities of the arc be confined to two curves whose equations are $y = F(x) = F$, $y = f(x) = f$. Then eliminating $\delta y_1$ by means of equations (2), Art. 76, (2) becomes, after omitting the common factor $dx_1$,

$$\left(y \pm \frac{r}{\sqrt{1+y'^2}} \pm \frac{rf'y'}{\sqrt{1+y'^2}}\right)_1 = 0, \tag{3}$$

and a similar equation for the lower limit. But since $ds$, any element of the arc, equals $\sqrt{1+y'^2}dx$, (3) may be written

$$y_1 \pm r\left\{\left(\frac{dx}{ds}\right)_1 + f_1'\left(\frac{dy}{ds}\right)_1\right\} = 0$$

$$= y_1 \pm r\left\{\cos m + \frac{\sin n}{\cos n}\sin m\right\}, \tag{4}$$

where $m$ is the angle which the tangent to the arc makes with the axis of $x$, and $n$ the angle which any tangent to the upper limiting curve makes with that axis. Let $t$ be the angle which the tangents to the arc and the limiting curve make with each other at the upper limit. Then, since $t$ is numerically equal to $n - m$, we have

$$\cos t_1 = \cos m_1 \cos n_1 + \sin m_1 \sin n_1. \tag{5}$$

Hence, clearing fractions, (4) gives

$$r \cos t_1 = y_1 \cos n_1,$$

and we can establish an equation of a similar character for the lower limit.

It must, however, be remembered that none of these results concerning variable limits can be confirmed as true maxima or minima without an examination of the terms of the second order, which examination would be impracticable.

## Problem XVI.

**98.** *It is required to determine the form of the solid of revolution which shall possess a given surface and a maximum volume, the generating curve being required to pass through two fixed points on the axis of revolution.*

Assume $x$ as the axis of revolution. Then the volume to be a maximum is $\int_{x_0}^{x_1} \pi y^2 dx$, while the given superficial area which must remain constant is $\int_{x_0}^{x_1} 2\pi y \sqrt{1 + y'^2}\, dx$. Hence, omitting the constant $\pi$, we have, by Euler's method, to maximize absolutely the expression

$$U = \int_{x_0}^{x_1} \left(y^2 + 2ay \sqrt{1 + y'^2}\right) dx = \int_{x_0}^{x_1} V dx. \tag{1}$$

Hence, after the usual transformation, we have

$$\delta U = \left(\frac{2ayy'}{\sqrt{1+y'^2}}\right)_1 \delta y_1 - \left(\frac{2ayy'}{\sqrt{1+y'^2}}\right)_0 \delta y_0$$

$$+ \int_{x_0}^{x_1} \left\{ 2y + 2a\sqrt{1+y'^2} - 2a\frac{d}{dx}\frac{yy'}{\sqrt{1+y'^2}} \right\} \delta y\, dx, \quad (2)$$

which equation is evidently true whether $\delta y_1$ and $\delta y_0$ vanish or not.

Here, as $V$ is a function of $y$ and $y'$ only, and $P = \frac{2ayy'}{\sqrt{1+y'^2}}$, we have by formula (C), Art. 56,

$$y^2 + \frac{2ay}{\sqrt{1+y'^2}} = c. \quad (3)$$

But since the generating curve is to meet the axis of $x$, $c$ must vanish, and we have

$$y^2 + \frac{2ay}{\sqrt{1+y'^2}} = y\left(y + \frac{2a}{\sqrt{1+y'^2}}\right) = 0. \quad (4)$$

Whence, if $y$ be not always zero, we have

$$y + \frac{2a}{\sqrt{1+y'^2}} = 0. \quad (5)$$

Hence

$$y'^2 = \frac{4a^2 - y^2}{y^2} \quad (6)$$

and

$$dx = \pm \frac{ydy}{\sqrt{4a^2 - y^2}}, \quad (7)$$

which, by integration, gives

$$x + b = \pm \sqrt{4a^2 - y^2}, \quad (8)$$

the equation of the circle whose radius is, numerically at least,

$2a$, and whose centre is on the axis of $x$, thus rendering the required solid a sphere.

**99.** We are evidently prevented, by the nature of this problem, from supposing that $y$ can ever become negative, and we may therefore use the positive sign only in equation (8). For if we were to regard $y$ as negative throughout any interval, say from $x_2$ to $x_3$, we would have the corresponding zone of surface negative, because $dx$ and $\sqrt{1+y'^2}$ are taken positively, which would be absurd. Hence we see from (5) that $2a$ is necessarily negative; and using its known numerical value, we have $2a = -r$.

**100.** We have now two constants, $r$ and $b$, to determine, since we were obliged to make $c$ vanish before we could completely integrate equation (3). But it will be observed that it would have been sufficient for a solution had we merely required the generating curve to meet the axis of $x$ at some point, taking this point as one of the limits, say the lower, and then regarding the limits as variable. By this method we would obtain a sphere, as before; and then if we impose the condition that both extremities of the generating curve shall be confined to the axis of $x$, as is most natural, we shall have a complete sphere. Hence, since the superficial area is given, $r^2$ is at once determined by dividing the area by $4\pi$, and the distance $x_1 - x_0$ being necessarily equal to $2r$, becomes also known; so that when one limiting value of $x$ is given, the other can be readily found. Now from (8) we see that $b$ is merely the abscissa of the centre of the circle or the sphere, and equals $x_0 + r$, or $x_1 - r$. As soon, therefore, as one of the limiting values of $x$ is given, all the required quantities can be determined; but if neither $x_0$ nor $x_1$ be given, $r$ only can be determined.

**101.** Thus far there would seem to be nothing peculiar or unsatisfactory about our solution; but we come now to speak

of a point which has occasioned considerable discussion among mathematicians, and which has led to an important extension of the calculus of variations.

Suppose that, as in the original enunciation of the problem, we require that $x_0$ and $x_1$ shall have assigned values, or that the value of $x_1 - x_0$ shall be assigned. Then the diameter of the sphere must be $x_1 - x_0$, and the only value which the surface of such a sphere can have is $\pi(x_1 - x_0)^2$, so that, as we are no longer at liberty to select a value for the superficial area, the solution appears at first to fail. But it has now been made apparent that the general solution of any problem of maxima or minima in the calculus of variations is entirely independent of any conditions which may be required to hold at the limits, the limits having been supposed fixed in the earlier problems for the sake of simplicity only. Therefore no general solution can be said to fail so long as it is always possible to assume such limiting values of $x, y, y'$, etc., as will satisfy all the conditions of the question which are necessarily involved in the general solution.

In the present case, if we require that the surface of the solid may be entirely generated by the revolving curve, these conditions are merely that the value of the superficial area may be assigned at pleasure, and that the generating curve shall have both extremities upon the axis of $x$, which conditions can, as we have seen, always be fulfilled by a sphere. Thus, since no restriction of the limits $x_0$ and $x_1$ is implied in the method by which the general solution was obtained, the apparent failure of the solution, when these limits are assigned, appears to arise from imposing too many conditions upon the question, some of which are incompatible, and for this the calculus of variations is evidently not responsible.

It will be remembered that in Prob. VII. we obtained as a general solution a catenary, having its directrix upon the axis of $x$, and then subsequently showed that the two fixed points could easily be so taken that no such catenary could be drawn.

In like manner, in Prob. XV. we shall be unable to draw the required arc if the given line be shorter than the right line which joins the two fixed points, or longer than a semicircumference constructed upon this right line as a diameter. In the first of these problems the conditions can, without changing the limiting values of $x$, always be satisfied by assuming suitable values for $y_0$ and $y_1$, and a similar remark will apply to the second problem unless the length of the given line be less than $x_1 - x_0$, in which case some change will become necessary in the limiting values of $x$ also.

The only peculiarity, then, about the present problem would seem to be that, while in the former two we are permitted to make various but not all possible assumptions regarding the quantities $x_1 - x_0$ and $y_1 - y_0$, here but one supposition regarding these quantities can be made for a given superficial area, and thus, as the probability of failure when we attempt beforehand to assign the limits, and also the surface, is vastly greater in this problem than in the other two, it more readily presents itself to our minds.

But we are naturally led to inquire whether there may not be some other solution for this and similar problems in those cases in which the general solution cannot be made applicable. This question, which has received much attention of late, and has led to an important extension in the calculus of variations, will be discussed in a subsequent section on *discontinuous solutions*. It will here be sufficient to say that such solutions do in many cases exist, and are generally composed of arcs of curves, or of right lines, or of some combination of both, and they are hence termed *discontinuous solutions*.

**102** Now if we put for $2a$ its value $-r$, the general equation given by the terms at the limits is

$$\left(y^2 - ry\sqrt{1+y'^2}\right)_1 dx_1 - \left(\frac{ryy'}{\sqrt{1+y'^2}}\right)_1 \delta y_1 = 0, \qquad (9)$$

together with a similar equation for the lower limit; and these equations are evidently like those of the preceding problem, except that they are multiplied by $y$, and $-r$ only is used. If we suppose $x_1$ and $x_0$ to be fixed, and $y_1$ and $y_0$ to be variable, (9) gives

$$\left(\frac{yy'}{\sqrt{1+y'^2}}\right)_1 = 0, \qquad \left(\frac{yy'}{\sqrt{1+y'^2}}\right)_0 = 0.$$

Hence we may have $y_1 = 0$, $y_0 = 0$, thus giving an entire sphere, which is satisfactory if the surface will permit. If one limiting value of $y$ be also given, the solution can always be effected, it being the closed segment of a sphere, having a given base and height, $r$ being determined by the equation

$$r^2 = \frac{\frac{s^2}{4\pi}}{s - \pi R^2}, \tag{10}$$

$s$ being the given surface, and $R$ the radius of the base. Regarding the other solution, $y_1' = 0$, $y_0' = 0$, it may be remarked that but one of these equations can ever be true, and therefore the other limit must be fixed.

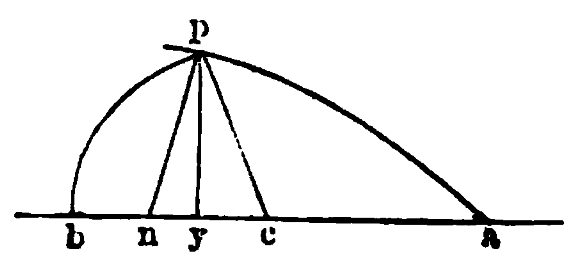
Now suppose the extremities of the generating curve to be limited to two other curves, all the curves being supposed to revolve about $x$, which is the same thing as limiting the sphere to two surfaces of revolution. Then, since the terms at the limits in this and the preceding problem compare as we have just shown, it will appear, by methods precisely like those employed in Art. 97, that we shall have

$$ry_1 \cos t_1 = y_1^2 \cos n_1, \tag{11}$$

together with a similar equation for the lower limit. Thus we have either $y_1 = 0$ and $y_0 = 0$, giving a complete sphere, or else the relation given in the last equation of Art. 97.

To interpret this relation, let $ap$ be the upper limiting curve,

$p$ the point of intersection with the arc whose centre is $c$, $py$ the ordinate $y_1$ of the limiting curve, and $np$ the normal.




Then $cpn = t_1$, and $npy = n_1$, and we have

$$\frac{r}{y_1} = \frac{\cos n_1}{\cos t_1}, \tag{12}$$

and this equation can only be satisfied by making $\cos t_1$ unity, which shows that the tangent to the limiting curve at $p$ must be parallel to the axis of $x$; that is, that $y_1$ must be either a maximum or minimum ordinate. But if $y_1$ should become equal to $r$, this relation would no longer be necessary, for then the lines $cp$ and $yp$ would coincide, the angles $cpn$ and $ypn$ become the same angle $cpn$, and (12) becomes merely $\frac{r}{r} = \frac{\cos cpn}{\cos cpn}$, which determines nothing regarding the direction of the normal or tangent to the limiting curve; and hence in this case the ordinate $y_1$ need be neither a maximum nor a minimum.

**103.** It must not, however, be assumed that all the results obtained in the last two articles will necessarily render the volume either a maximum or a minimum. For we have already seen that it is always necessary to appeal to the terms of the second order before the results obtained by making those of the first vanish can be interpreted. We have, moreover, also stated that the discussion of these limiting terms, when the general solution is known, is a problem of the differential calculus rather than of the calculus of variations, and particularly so when the terms of the second order are to be considered. As a means of illustrating both these remarks,

we shall consider only the case in which one limiting value of $y$ is zero, and take the liberty, as that work is now inaccessible to most readers, of copying the discussion from Todhunter's History of Variations, p. 408.

Let $y$ be any ordinate of the limiting curve, $h$ the height of the segment, $v$ its volume, and $s$ its surface. Then, since the segment is known from the general solution obtained from variations to be always spherical in form, and by supposition has but one base, we have, $r$ being the radius of the sphere,

$$v = \pi\left(rh^2 - \frac{h^3}{3}\right), \tag{1}$$

and we can now, by the differential calculus, determine the conditions which will render $v$ a maximum or a minimum, supposing $s$ to remain constant. Since $s = 2\pi rh$ is to remain constant, $rh$ is a constant, say $k^2$. Then from the equation of the circle, when the origin is at the extremity of any diameter, we have

$$y^2 = 2rh - h^2 = 2k^2 - h^2;$$

whence

$$h^2 = 2k^2 - y^2,$$

and therefore (1) becomes

$$v = \pi\left\{k^2\sqrt{2k^2-y^2} - \frac{\sqrt{(2k^2-y^2)^3}}{3}\right\}. \tag{2}$$

Whence

$$\frac{dv}{dy} = \pi\left\{\frac{-k^2y}{\sqrt{2k^2-y^2}} + y\sqrt{2k^2-y^2}\right\} = \frac{\pi y(k^2-y^2)}{\sqrt{2k^2-y^2}}; \tag{3}$$

and since the differential of the limiting curve must be $dy = y'dx$, we have

$$\frac{dv}{dx} = \frac{\pi yy'(k^2-y^2)}{\sqrt{2k^2-y^2}}. \tag{4}$$

To make the second member of this equation vanish, we must have $y' = 0$, $y = k$, or $y = 0$.

To test these solutions, write $u = k^2 - y^2$, $z = \sqrt{2k^2 - y^2}$. Then

$$\frac{d^2v}{dx^2} = \frac{\pi}{z^3}\left(z^2uyy'' + z^2uy'^2 - 2z^2y^2y'^2 + uy^2y'^2\right). \qquad (5)$$

Whence it readily appears that if $y'$ vanish, making $y$ a maximum or a minimum ordinate according as $y''$ is negative or positive, $v$ will have the like or contrary property to $y$ according as $u$ is positive or negative.

If $y = k$, without making $y'$ vanish—that is, without being at the same time a maximum or aminimum ordinate—$\frac{d^2v}{dx^2}$ will be negative, and $v$ will be a maximum. But if $y$, while equal to $k$, be also a maximum or a minimum ordinate—that is, make $y'$ vanish—$\frac{d^2v}{dx^2}$ will also vanish, and it will be found by trial that the third differential will do so likewise, while $\frac{d^4v}{dx^4}$ will become negative or positive according as $y$ is a maximum or a minimum, thus making $v$ have the like maximum or minimum property with $y$.

We have already seen that the question does not permit us to suppose that $y$ can become negative, and hence the limiting curve must be such that when $y$ is zero it shall be a minimum ordinate, which will cause $y'$ to become zero also. These suppositions will render $\frac{d^4v}{dx^4}$ positive, having reduced the preceding differential coefficients to zero. Therefore the supposition that $y$ is zero renders $v$ a minimum.

The foregoing results, which have been verified by the author, appear to be correct, although they do not agree with those obtained by Prof. Jellett. (See his page 165.)

We have not yet examined the terms of the second order in the general solution obtained by the calculus of variations in the problem as originally given, but shall resume this point hereafter.

**104.** It will be remembered that we were unable to integrate equation (3), Art. 98 (that is, the equation $M = 0$), without supposing $c$ to become zero. Nevertheless this differential equation has been shown to be that of a curve traced by the focus of some conic section as it is rolled along the axis of $x$, and the following outline of the demonstration is, with some difference of notation, given by Prof. Jellett on page 364, but the proof is due to Delaunay.

Let $r = f(v) = f$ be the polar equation of any curve, the pole being assumed at pleasure; and when that curve is rolled along the axis of $x$, let $y = F(x) = F$ be the equation of the curve traced by that pole. Then the following relations are not difficult to establish:

$$y' = \frac{dr}{rdv}, \tag{1}$$

$$y = \frac{r^2 dv}{\sqrt{dr^2 + r^2 dv^2}}. \tag{2}$$

By means of these relations we are sometimes able, when the equation (differential or other) of one curve is known, to determine that of the other; and such is the case in the present instance. Now write equation (3), Art. 98, thus:

$$by = (y^2 + d)\sqrt{1 + y'^2}, \tag{3}$$

where $b = -2a$ and $d = -c$. Then, from (1), we obtain

$$\sqrt{1 + y'^2} = \frac{\sqrt{dr^2 + r^2 dv^2}}{rdv}. \tag{4}$$

Substituting in (3) the values of $y$ and $\sqrt{1+y'^2}$ from (2) and (4), we obtain

$$dv = \frac{\sqrt{d}\,dr}{r\sqrt{br - r^2 - d}}. \tag{5}$$

The integral of this equation is known to give

$$\frac{1}{r} = \frac{b}{2d} - \sqrt{\frac{b^2}{4d^2} - \frac{1}{d}}\cos v$$

$$= \frac{A}{d} - \sqrt{\frac{A^2}{d^2} - \frac{1}{d}}\cos v, \tag{6}$$

where $A = \frac{b}{2}$. If now we assume, as the polar equation of the conic section,

$$\frac{1}{r} = \frac{1 + e\cos v}{A(1 - e^2)}, \tag{7}$$

we can obtain from it equation (6) by merely making $e$ equal to $\sqrt{1 - \frac{d^2}{A^2}}$, and hence the truth of the proposition is established.

The curves which may be thus described are exceedingly various. Thus, if we make $d = -c$ vanish, the conic section will become a straight line, and the curve generated will be a circle, giving a sphere as a general solution, which agrees with what has been already shown. Moreover, the circle is evidently the only one of these curves which can ever meet the axis of $x$. Again, if we take the circle as the conic section, the curve, traced by its focus—that is, its centre—will be a right line parallel to the axis of $x$, and the required solid will be a cylinder.

### Problem XVII.

**105.** *It is required to determine the form which a uniform cord of given length, whose extremities are confined to two fixed points or curves, must assume in order that its centre of gravity may be at a maximum depth.*

Take the horizontal as the axis of $x$, and let $L$ or

$$\int_{x_0}^{x_1} \sqrt{1+y'^2}\,dx$$

be the given length of the cord, which, by the conditions of the problem, is to remain constant. Then, by the well-known principles of finding the co-ordinates of the centre of gravity of any curve, we shall have, $D$ being the depth, which is to become a maximum,

$$D = \frac{1}{L}\int_{s_0}^{s_1} y\,ds = \frac{1}{L}\int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx.$$

Hence, by Euler's method, we are to maximize absolutely the expression

$$U = \int_{x_0}^{x_1}\left\{\frac{y\sqrt{1+y'^2}}{L} + a\sqrt{1+y'^2}\right\}dx = \int_{x_0}^{x_1} V\,dx.$$

Here $V$ is a function of $y$ and $y'$ only, and

$$P = \frac{yy'}{L\sqrt{1+y'^2}} + \frac{ay'}{\sqrt{1+y'^2}}, \tag{1}$$

so that by formula (C), Art. 56, we have

$$\frac{y\sqrt{1+y'^2}}{L} + a\sqrt{1+y'^2} = \frac{yy'^2}{L\sqrt{1+y'^2}} + \frac{ay'^2}{\sqrt{1+y'^2}} + c. \tag{2}$$

Whence, by reduction, we obtain

$$\frac{y + aL}{\sqrt{1 + y'^2}} = cL, \tag{3}$$

and

$$1 + y'^2 = \frac{(y + aL)^2}{c^2 L^2}, \tag{4}$$

which, to be rendered integrable, must be solved thus:

$$dx = \pm \frac{cLdy}{\sqrt{(y + aL)^2 - c^2 L^2}}. \tag{5}$$

Integrating this equation, we obtain

$$x = Al\left(y + B + \sqrt{(y + B)^2 - A^2}\right) + C, \tag{6}$$

where $A = cL$, $B = aL$. Comparing this equation with equation (5), Art. 59, we see that it is also the equation of a catenary, in which $y + B$ is put for $y$; because the reasoning in Art. 59 will apply equally to any curve whose equation is of that form, and this equation will take that form if, while keeping the axis of $x$ horizontal, we remove it so as to make $B$ zero. Indeed, without integrating, we may at once reach this conclusion. For by comparing equation (3) with equation (2), Art. 59, we see it to be the differential equation of a catenary, as described.

**106.** To determine the constants $A, B, C$, we have the conditions that the curve must, if its extremities be fixed, pass through those fixed points, and must have also a given length, and these three conditions are sufficient, assuming that we can solve any exponential equation which may arise. Comparing (6) with equation (5), Art. 59, we see that if we make the axis of $x$ pass through one of the given points, and estimate $y$ upward, $B$ will be the distance from the axis of $x$ to

the directrix, estimated positively; but if we estimate $y$ downward, $B$ will have the same numerical value, but will be negative. We adopt, however, the former supposition. Then, as $L$ is positive, $a$ or $\frac{B}{L}$ must be also taken positively. We may, if we choose, dispose of the constant $C$, as we did of the constant $b$ in Art. 59, by making it $-AlA$.

If, then, we can determine $A$, the discussion of the constants will be complete; and this may be done in the following manner: Let $D$ denote $x_1 - x_0$, which is supposed to be known, and $E$, $y_1 - y_0$, which is also known, and let the ordinate, drawn to the lowest point of the catenary, divide $D$ into two segments, $f$ and $g$, while the corresponding segments of the arc $L$ are $m$ and $n$, so that we have

$$f + g = D, \tag{7}$$

$$m + n = L. \tag{8}$$

Then, in discussions of the catenary, the following equation is easily established:

$$m = \frac{A}{2}\left(e^{\frac{f}{A}} - e^{-\frac{f}{A}}\right), \tag{9}$$

together with a similar equation between $g$ and $n$. Whence

$$L = \frac{A}{2}\left(e^{\frac{f}{A}} - e^{-\frac{f}{A}} + e^{\frac{g}{A}} - e^{-\frac{g}{A}}\right). \tag{10}$$

Now because the catenary passes through the two fixed points, we have from its equation, (10) of Art. 59,

$$E = \frac{A}{2}\left(e^{\frac{f}{A}} + e^{-\frac{f}{A}} - e^{\frac{g}{A}} - e^{-\frac{g}{A}}\right), \tag{11}$$

which equation, combined with the preceding four, will evidently determine $A$, which in statics denotes numerically the tension which the cord will sustain at its lowest point.

**107.** If the extremities be not fixed, but merely confined to two curves, the general solution will of course be unchanged, only certain conditions must hold at the limits. For now the limiting terms, which vanished when the extremities were fixed, become

$$V_1 dx_1 - V_0 dx_0 + P_1 \delta y_1 - P_0 \delta y_0 = 0. \tag{12}$$

Substituting the values of $P$ and $V$ from (1) and the preceding equation, (12) gives, for the upper limit,

$$\left(\frac{y}{L} + a\right)_1 \sqrt{(1 + y'^2)_1}\, dx_1 + \left(\frac{y}{L} + a\right)_1 \left(\frac{y'}{\sqrt{1 + y'^2}}\right)_1 \delta y_1 = 0, \tag{13}$$

together with a similar equation for the lower limit. Let the equation of the upper limiting curve be $dy = f'dx$. Then eliminating $\delta y_1$ by the equation

$$\delta y_1 = (f' - y')_1 dx_1, \tag{14}$$

(13) gives

$$\left(\frac{y + aL}{L\sqrt{1 + y'^2}}\right)_1 (1 + y_1' f_1') = 0. \tag{15}$$

Now, to make the first factor vanish, we must have either $y = -aL = -B$ or $y' = \pm \infty$. But since $B$ is numerically equal to the distance of the directrix from the axis of $x$, this supposition would make the lowest point of the cord touch the directrix, and this could not be unless the tension were zero, in which case the cord would hang in a double right line, having its extremities at a common point. Neither can

we suppose $y_1'$ infinite. For, from the general equation of the catenary, we have, by differentiation,

$$y' = \frac{1}{2}\left(e^{\frac{x}{A}} - e^{-\frac{x}{A}}\right); \tag{16}$$

and to make this infinite we must have $\frac{x}{A}$ infinite, giving either $A$ zero, which condition has just been discussed, or $x_1$ zero, which would make the catenary a right line as before.

Hence we must have $1 + y_1'f_1' = 0$, and a similar condition will evidently hold if the other extremity be confined to another limiting curve. Therefore we conclude that the catenary will cut its limiting curves at right angles, the constants in this case being determined by the conditions that the catenary must have a given length, and that its extremities must cut two given curves at right angles, or pass through a fixed point and cut one given curve at right angles.

**108.** The terms of the second order in the case in which the extremities are fixed are

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \frac{y + B}{L\sqrt{(1 + y'^2)^3}}\,\delta y'^2\,dx. \tag{17}$$

Now from equation (3) we see that $y + B$ must be of the same sign as $c$. Now writing (3) $y + B = c\sqrt{1 + y'^2}$, differentiating and dividing by $dy$ or $y'dx$, we obtain

$$\frac{cy''}{\sqrt{1 + y'^2}} = 1, \tag{18}$$

and therefore $c$, and consequently $y + B$, must be of the same sign as $y''$. Now if we estimate $y$ upward, the catenary is convex to the axis of $x$, and $y''$, and therefore $y + B$, is positive, and we have a minimum. But if we estimate $y$ downward, $y + B$ is negative, and we have a maximum.

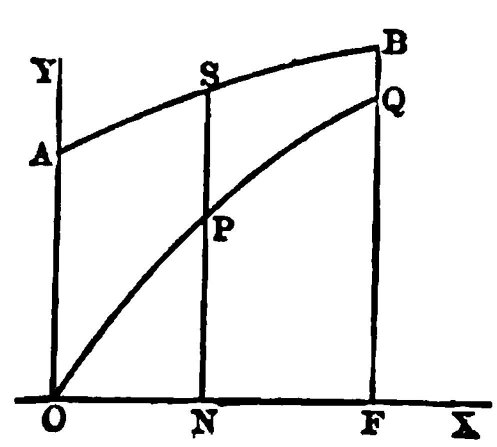
These different results appear to be due to the fact that

by estimating $y$ upward we make the distance of the centre of gravity from the axis of $x$ approach as near as possible to $-\infty$, while by estimating $y$ downward, we make it approach as near as possible to $+\infty$, its numerical value in either case being the same, and a maximum.

**109.** If we assume the vertical as the independent variable, the general solution must be the same whether we can obtain it by that method or not. For whatever change can be made in the form of the required curve by ascribing variations to $y$ and its differential coefficients with respect to $x$, can, at least if the curve be continuous and drawn between fixed points, also be made by ascribing suitable variations to $x$ and its differential coefficients with respect to $y$, $y$ itself receiving no variation. This principle will be found to aid us in the solution of the following problem.

### Problem XVIII.*

**110.** *It is required to draw between two fixed points A and B a curve of given length having the following property: that if at any point S of the required curve an ordinate NS be drawn, and on it we lay off NP equal to the arc AS, the curve traced by the point P shall enclose a maximum or minimum area.*




* This problem is only a particular case of the second of the celebrated isoperimetrical problems given by James Bernoulli, the original problem requiring $NP$ to be any function of the arc $AS$, which can, of course, not be fully solved so long as the nature of the function is entirely undetermined. The solution is from the Adams Essay, Chapter XI.

Here the area to be made a maximum or a minimum is $\int_{x_0}^{x_1} s dx$, $s$ being the length of the arc measured from $A$, while $\int_{x_0}^{x_1} \sqrt{1+y'^2}dx$ (that is, the length of the arc $ASB$) is to remain constant. Hence we are, by Euler's method, to maximize or minimize the expression

$$U = \int_{x_0}^{x_1} \left(s + a\sqrt{1+y'^2}\right) dx. \tag{1}$$

Hence, to the second order,

$$\delta U = \int_{x_0}^{x_1} \left\{ \delta s + \frac{ay'}{\sqrt{1+y'^2}} \delta y' + \frac{a}{2\sqrt{(1+y'^2)^3}} \delta y'^2 \right\} dx. \tag{2}$$

But

$$s = \int_{x_0}^{x} \sqrt{1+y'^2}\, dx.$$

Whence

$$\delta s = \int_{x_0}^{x} \frac{y'}{\sqrt{1+y'^2}} \delta y'\, dx + \frac{1}{2}\int_{x_0}^{x} \frac{1}{\sqrt{(1+y'^2)^3}} \delta y'^2\, dx, \tag{3}$$

and

$$\int \delta s\, dx = x\delta s - \int x \frac{d\delta s}{dx} dx. \tag{4}$$

Hence, taking this integral from $x_0$ to $x_1$, and observing from the figure that $x_0$ is zero, while $b$ may be put for $x_1$, because it is constant, we have

$$\int_0^b \delta s\, dx = b\int_0^b \left\{ \frac{y'}{\sqrt{1+y'^2}} \delta y' + \frac{1}{2\sqrt{(1+y'^2)^3}} \delta y'^2 \right\} dx$$

$$- \int_0^b x \left\{ \frac{y'}{\sqrt{1+y'^2}} \delta y' + \frac{1}{2\sqrt{(1+y'^2)^3}} \delta y'^2 \right\} dx$$

$$= \int_0^b (b-x) \left\{ \frac{y'}{\sqrt{1+y'^2}} \delta y' + \frac{1}{2\sqrt{(1+y'^2)^3}} \delta y'^2 \right\} dx. \tag{5}$$

Substituting this value in (2), and employing the usual notation for the limits, we have

$$\delta U = \int_{x_0}^{x_1} (a + b - x) \left\{ \frac{y'}{\sqrt{1 + y'^2}} \delta y' + \frac{1}{2\sqrt{(1 + y'^2)^3}} \delta y'^2 \right\} dx. \quad (6)$$

Now examining the second factor of the second member of (6), we see that it is the variation of $\sqrt{1 + y'^2}\, dx$, or $ds$, and that $b - x$, or $Z$, is the distance of any point of the arc $ASB$ from the line $BF$, and therefore it is not difficult to see that the problem is really in solution as though, taking the vertical as the independent variable, we had required the form of the curve of given length passing through $A$ and $B$, and having the distance of its centre of gravity from $BF$ a maximum or a minimum. Therefore, without solving (6) in detail, we know from the last article of the preceding problem that this curve is a catenary, having its directrix parallel to the axis of $y$.

**111.** But some investigation will be necessary in order to determine the sign of the terms of the second order. For although, as before, it is evident that $a$, like $B$ of the last problem, is numerically equal to the perpendicular distance from $BF$ to the directrix, its sign is not at once clear. Treating the terms of the first order in (6) in the usual way, we obtain

$$(a + b - x) \frac{y'}{\sqrt{1 + y'^2}} = c. \quad (7)$$

Whence

$$a + b - x = \frac{c\sqrt{1 + y'^2}}{y'}. \quad (8)$$

Differentiating (8), and dividing by $dx$, we have

$$\frac{cy''}{y'^2\sqrt{1 + y'^2}} = 1, \quad (9)$$

from which it appears that $c$ must always be of the same sign as $y''$. But the catenary may be either convex or concave to the axis of $x$, so that $c$ will be positive in the former and negative in the latter case. Moreover, we see from (7) that $a + b - x$ must always be of the same sign as $c$, and therefore the terms of the second order will become positive when the catenary is convex to the axis of $x$, and negative when it is concave, thus showing that the area in question will be a minimum in the former and a maximum in the latter case.

## Problem XIX.

**112.** *It is required to determine the form of the solid of revolution of given mass and uniform density which will exert a maximum attractive force upon a particle situated upon the axis of revolution.*

Take the axis of revolution as that of $x$, and let the attracted particle be at the origin. Moreover, conceive the solid to be divided into slices of the thickness $dx$, by planes perpendicular to $x$. Then, by dividing these slices into differential rings, it is easily found that, omitting the factor of density, because constant, the force exerted by any slice in the direction of $x$ is

$$2\pi \left\{ 1 - \frac{x}{\sqrt{x^2 + y^2}} \right\} dx.$$

Hence

$$\int_{x_0}^{x_1} \left\{ 1 - \frac{x}{\sqrt{x^2 + y^2}} \right\} dx$$

is to be a maximum, while the volume $\int_{x_0}^{x_1} \pi y^2 dx$ is to remain

constant. Hence, by Euler's method, we maximize the expression

$$U=\int_{x_0}^{x_1}\left\{1-\frac{x}{\sqrt{x^2+y^2}}+ay^2\right\}dx=\int_{x_0}^{x_1}Vdx. \tag{1}$$

Therefore, to the second order, we have

$$\delta U=\int_{x_0}^{x_1}\left\{\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}+2ay\right\}\delta y\,dx$$

$$+\int_{x_0}^{x_1}\left\{a+\frac{x(x^2-2y^2)}{2(x^2+y^2)^{\frac{5}{2}}}\right\}\delta y^2 dx. \tag{2}$$

Here $V$ is a function of $x$ and $y$ only, and the terms of the first order in $\delta U$ need no transformation; so that we have at once, unless $y$ be always zero,

$$\frac{x}{(x^2+y^2)^{\frac{3}{2}}}+2a=0. \tag{3}$$

Now putting $-\frac{1}{c^2}$ for $2a$, (3) gives

$$(x^2+y^2)^{\frac{3}{2}}=c^2x. \tag{4}$$

But $v$, the volume, or $\int_{x_0}^{x_1}\pi y^2 dx$, is a known quantity, and (4) gives

$$y^2=(c^2x)^{\frac{2}{3}}-x^2. \tag{5}$$

Whence

$$v=\pi\int_{x_0}^{x_1}(c^{\frac{4}{3}}x^{\frac{2}{3}}-x^2)\,dx. \tag{6}$$

But the general integral of the second member of (6) is

$$\pi\left(\frac{3}{5}c^{\frac{4}{3}}x^{\frac{5}{3}}-\frac{1}{3}x^3\right)+d. \tag{7}$$

Now suppose $x_0$ to be zero, which will place the particle upon the surface of the solid; and assume also that when $x = x_1$ the generating curve meets the axis of $x$. Then, by making $y$ zero, and $x$, $x_1$, in (4), we see that $x_1$ is numerically equal to $c$; and taking (7) between the limits 0 and $c$, we find

$$v = \frac{4\pi c^3}{15}. \tag{8}$$

It therefore appears that, when the volume is given, the length of the axis is not in our power, but is determined by that volume; and $c^2$ being known, $a$ is also known.

**113.** Now the coefficient of $\delta y^2\, dx$ in (2) is

$$a + \frac{x(x^2 - 2y^2)}{2(x^2 + y^2)^{\frac{5}{2}}}. \tag{9}$$

Putting for $a$ its value $-\frac{1}{2c^2}$, and substituting for the first members of (4) and (5) the second members of the same equations, (9) becomes

$$-\frac{1}{2c^2} + \frac{x(3x^2 - 2c^{\frac{4}{3}} x^{\frac{2}{3}})}{2(c^2x)^{\frac{5}{3}}}, \quad \text{or} \quad -\frac{1}{2c^2} + \frac{3x^{\frac{4}{3}} - 2c^{\frac{4}{3}}}{2c^{\frac{10}{3}}},$$

$$\text{or} \quad \frac{3(x^{\frac{4}{3}} - c^{\frac{4}{3}})}{2c^{\frac{10}{3}}}. \tag{10}$$

Hence the terms of the second order become

$$\delta U = \frac{3}{2c^{\frac{10}{3}}} \int_{x_0}^{x_1} (x^{\frac{4}{3}} - c^{\frac{4}{3}})\, \delta y^2\, dx = E \int_{x_0}^{x_1} Z\, \delta y^2 dx. \tag{11}$$

Now, since $v$ cannot be negative, we see from (8) that $c$ must be positive, and it is numerically greater than $x$, being equal to $x_1$. Therefore $E$ is positive, while $Z$ can never become positive, and the terms of the second order become invariably negative, thus giving us a maximum.

## Problem XX.*

**114.** *It is required to determine the form of the solid of revolution, having a given base and volume, which will experience a minimum resistance in passing through a fluid in the direction of the axis of revolution.*

Let $x$ be the axis of revolution. Then, reasoning as in Prob. VI., we see that we must minimize $\int_{x_0}^{x_1} \frac{yy'^3}{1+y'^2}\,dx$, while, the volume being given, we must have $\int_{x_0}^{x_1} y^2 dx$ constant. Therefore, by Euler's method, we minimize absolutely the expression

$$U = \int_{x_0}^{x_1} \left\{ \frac{yy'^3}{1+y'^2} + 2ay^2 \right\} dx = \int_{x_0}^{x_1} V dx. \tag{1}$$

Here $V$ is a function of $y$ and $y'$, and

$$P = \frac{y(3y'^2 + y'^4)}{(1+y'^2)^2}, \tag{2}$$

so that by formula (C), Art. 56, we have

$$\frac{yy'^3}{1+y'^2} + 2ay^2 = \frac{yy'(3y'^2 + y'^4)}{(1+y'^2)^2} + b. \tag{3}$$

We will assume that the generating curve cuts the axis of $x$, which will render $b$ zero, and then we easily obtain the equations

$$ay = \frac{y'^3}{(1+y'^2)^2} \qquad \text{and} \quad y = \frac{cy'^3}{(1+y'^2)^2}, \tag{4}$$

$c$ being put for $\frac{1}{a}$.

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* The following discussion, which is much more satisfactory than that of Prob. VI., appears to be almost entirely due to Prof. Todhunter. (See his Adams Essay, Chapter X., from which this solution is taken.)

**115.** Now the last equation can be shown to indicate that the generating curve is a hypocycloid. For let $y' = \tan v$. Then, by (4), we have

$$y = c \sin^3 v \cos v, \tag{5}$$

and, by differentiation, we have

$$y' = c(3 \sin^2 v \cos^2 v - \sin^4 v)\frac{dv}{dx}$$

$$= c(3 \cos^2 v - \sin^2 v) \sin^2 v \frac{dv}{dx}. \tag{6}$$

Hence

$$\frac{dy}{dv} = c(3 \cos^2 v - \sin^2 v) \sin^2 v. \tag{7}$$

Dividing (7) by $y' = \tan v = \dfrac{\sin v}{\cos v}$, we have

$$\frac{dx}{dv} = c(3 \cos^2 v - \sin^2 v) \sin v \cos v. \tag{8}$$

Squaring and adding (7) and (8), observing that

$$\sin^4 v + \sin^2 v \cos^2 v = \sin^2 v (\sin^2 v + \cos^2 v) = \sin^2 v . 1,$$

we have, putting $ds$ for an element of the arc,

$$\frac{ds}{dv} = c(3 \cos^2 v - \sin^2 v) \sin v = c \sin 3v. \tag{9}$$

To integrate, write this equation thus:

$$ds = \frac{c}{3} \sin 3v \, d(3v).$$

Then we obtain

$$s = -\frac{c}{3} \cos 3v + c_1. \tag{10}$$

This equation is known to indicate that the curve is a hypocycloid, the radius of the rolling circle being one third that of the fixed circle. If now we suppose that when $y$ vanishes $v$ vanishes also, and measure $s$ from this point, we have

$0 = -\frac{c}{3}\cos 0 + c_1$; that is, $c_1 = \frac{c}{3}$; and (10) becomes

$$s = \frac{c}{3}(1 - \cos 3v). \tag{11}$$

**116.** To determine the constant $c$, we have the conditions that the solid must have a given base and an assigned volume, and we may use these conditions thus: Let $v_1$ be what $v$ becomes when $x = x_1$ and when $y = y_1$, a known constant, say $B$. Then it is shown that the volume of the solid is

$$\frac{\pi B^3\left(\frac{3}{8} - \frac{7}{10}\sin^2 v_1 + \frac{1}{3}\sin^4 v_1\right)}{\sin v_1 \cos^3 v_1}. \tag{12}$$

We have also, from (5),

$$B = c\sin^2 v_1 \cos v_1; \tag{13}$$

and from these equations $v_1$ and $c$ may be determined.

This solution, however, like some others, is not always possible. For it is shown that the volume can be as great as we please, but that it diminishes as $v_1$ increases, and has its least value when $v_1 = \frac{\pi}{3}$, its value then being $\frac{\pi B^3\sqrt{3}}{5}$. If, therefore, the given volume be less than this quantity, no such solid, with the given volume, could be constructed upon the given base.

**117.** Let us now examine the terms of the second order. We may evidently write $U$ thus:

$$U = \int_{x_0}^{x_1}(yf + 2ay^2)\,dx, \tag{14}$$

where $f = \frac{y'^2}{1+y'^2}$, and is therefore a function of $y'$ only. Hence the terms of the second order arising from the expression $\int_{x_0}^{x_1} yf dx$ may be treated as in Prob. VIII., while the term arising from $\int_{x_0}^{x_1} 2ay^2 dx$ is evidently $\frac{1}{2}\int_{x_0}^{x_1} 4a\delta y^2 dx$. Therefore, by the formula of Prob. VIII., we have, when we suppose the limits to be fixed,

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \left\{ f''(y\delta y'^2 - y''\delta y^2) + 4a\delta y^2 \right\} dx, \tag{15}$$

where

$$f' = \frac{df}{dy'} = \frac{3y'^2 + y'^4}{(1+y'^2)^2}, \tag{16}$$

and

$$f'' = \frac{d^2 f}{dy'^2} = \frac{2y'(3 - y'^2)}{(1+y'^2)^3}. \tag{17}$$

But, from the first equation (4), we have

$$ay' = \frac{y''(3y'^2 - y'^4)}{(1+y'^2)^3}, \qquad \text{whence} \quad 2a = f''y''. \tag{18}$$

Substituting this value in (15), we have

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} f''(y\delta y'^2 + y''\delta y^2)\, dx. \tag{19}$$

Now $f''$ is positive so long as $y'^2$ does not exceed three; that is, when $v$ does not exceed $\frac{\pi}{3}$; and $y''$ is here positive also, so that the integral becomes positive.

**118.** But since the distance $x_1 - x_0$ is not fixed, it is evident that the limits of integration are not altogether fixed. But as the base is given, and we may consider its distance from the origin as fixed, the limit $x_1$ may be regarded as fixed, as is also $y_1$. Now the terms of the first and second order arising from the variation of $x_0$ and $y_0$ evidently are

$$-V_0\,dx_0 - P_0\delta y_0 - \frac{1}{2}\left(\frac{dV}{dx}\right)_0 dx_0^2 - \delta V_0\,dx_0 - \frac{1}{2}f_0'\,\delta y_0^2, \quad (20)$$

the last term resulting from the formula in Prob. VIII., when $\delta y$ at either limit does not vanish. But

$$P_0 = y_0 f_0', \qquad \frac{dV}{dx_0} = y_0 f_0' y_0'' + f_0 y_0' + 4ay_0 y_0',$$

$$\delta V_0 = y_0 f_0'\,\delta y_0' + f_0\,\delta y_0 + 4ay_0\,\delta y_0.$$

Now since $y_0$ is zero, and, as appears from (4), $y_0'$ is also zero, all the quantities $V_0, f_0, f_0', \dfrac{dV}{dx_0}$ will separately vanish, and the terms in (20) will disappear. Therefore the variation arising from a change in $x_0$ and $y_0$ is not even of the second order, although it might still be a quantity of the third order; and as the integral in (19) is positive, we have in this case a solid of minimum resistance.

## Problem XXI.

**119.** *Let a curve meet the axis of $x$ at two fixed points, the origin being assumed midway between them. Then it is required to determine the form of this curve, so that, being revolved about the axis of $x$, it may generate a solid of given volume whose moment of inertia, with respect to the axis of $y$, may be a minimum.*

Conceive the solid to be divided into slices by planes perpendicular to the axis of $x$. Then the moment of inertia of any slice, whose thickness is $dx$, is

$$\pi m\left(\frac{y^4}{4}+x^2y^2\right)dx, \tag{1}$$

where $m$ denotes the mass, which is constant. This equation is easily obtained from the moment of inertia of the rings of which the slice is composed, which is

$$M\left(\frac{y^2}{2}+x^2\right), \tag{2}$$

$M$ being the mass of the ring, or $2\pi m\, dy\, dx$. Therefore, since the volume is to remain constant, we must, by Euler's method, minimize the expression

$$U=\int_{x_0}^{x_1}\left\{\frac{y^4}{4}+x^2y^2-a^2y^2\right\}dx=\int_{x_0}^{x_1}V dx. \tag{3}$$

Of course we could have put $a$ for $-a^2$ as the indeterminate multiplier, and this is what we would naturally do in first investigating the problem; still the present form is known to be more convenient.

Now we have

$$\delta U=\int_{x_0}^{x_1}(y^3+2x^2y-2a^2y)\delta y\, dx=\int_{x_0}^{x_1}y(y^2+2x^2-2a^2)\delta y\, dx. \tag{4}$$

Hence, if $y$ be not always zero, we have

$$y^2+2x^2=2a^2, \tag{5}$$

which shows that the solid must be an oblate spheroid in which the major axis is to the minor as $\sqrt{2}$ is to unity.

**120.** The terms of the second order are

$$\delta U = \int_{x_0}^{x_1} \left\{ \frac{3y^2}{2} + x^2 - a^2 \right\} \delta y^2 dx,$$

which, by means of (5), reduce to $\delta U = \int_{x_0}^{x_1} y^2 \delta y^2 dx$, and this being necessarily positive, we have a minimum.

But while the solution is thus apparently satisfactory, it evidently affords another example of the kind discussed in Prob. XVI. For if we suppose the limits $x_0$ and $x_1$ to be assigned—that is, the minor axis of the ellipse—then, unless the volume be just $\frac{8\pi B^3}{3}$, in which $B$ is the semi-minor axis, no such spheroid can be constructed. But if, without assigning the limits except to make the curve meet the axis of $x$ at two points equally distant from the origin, we only require the figure into which a given volume must be formed, as above, we shall obtain a spheroid in which the axes are related as just mentioned, the limiting values of $x$ having been determined by the given volume. Still, in the investigation of the terms of the second order just given, we have assumed that $x_1$ and $x_0$ undergo no change. Nevertheless, if we vary $x$ and $y$ at the limits, we shall not increase these terms, since, $y$ at the limits being zero, $V_1$, $V_0$, $\delta V_1$, $\delta V_0$, $\left(\frac{dV}{dx}\right)_1$ and $\left(\frac{dV}{dx}\right)_0$ severally vanish.

Here the constants are all determined by the assigned volume, combined with the conditions that $y_0$ and $y_1$ shall be zero. For $B$ is determined from the condition that the volume $\frac{8\pi B^3}{3}$ must equal an assigned quantity; then $A$, the semi-major axis, by the known relation between the axes; after which $a^2$ is found by means of (5).

## SECTION VII.

### *CASE IN WHICH V IS A FUNCTION OF POLAR CO-ORDINATES AND THEIR DIFFERENTIAL COEFFICIENTS.*

**121.** The principles of the calculus of variations thus far obtained are equally applicable when polar co-ordinates are to be employed; and as the mode of applying these principles is precisely similar to that which we have already given for rectangular co-ordinates, we shall present but two examples, the first of absolute, and the second of relative maxima and minima.

### Problem XXII.

*A particle which is always attracted towards a fixed centre, with a force which varies according to the Newtonian law of gravity, is projected from a fixed point so as to pass through a second fixed point. It is required to determine the nature of its path, assuming that it must be the path of least or minimum action.*

Assume the attracting centre as the pole, $r$ as the radius vector, or distance of the particle at any time, from the centre of force, $r_0$ and $r_1$ the distance of the first and second points respectively, and $v$ the natural angle included between $r_0$ and any other radius vector. Also let $f$, a constant, be the intensity of the force at a unit's distance, and $v'$ the velocity of the particle in its orbit at any instant.

Now, from mechanics, the action of the path is

$$\int_{s_0}^{s_1} v'ds, \tag{1}$$

where $ds$ is an element of the path. But

$$ds = \sqrt{dr^2 + r^2dv^2} = dv\sqrt{r^2 + \frac{dr^2}{dv^2}} = \sqrt{r^2 + r'^2}dv, \tag{2}$$

so that the action becomes

$$\int_{v_0}^{v_1} v' \sqrt{r^2 + r'^2}\, dv. \tag{3}$$

Now in determining $v'$ three cases arise. For we know that the path of a revolving particle will be an ellipse, a parabola, or an hyperbola, according as $v_0'$, the velocity of projection, is less, equal to, or greater than $\sqrt{\frac{2f}{r_0}}$. Let us here consider the first case, and suppose $v_0' = \sqrt{\frac{2f}{r_0} - \frac{f}{a}}$. Then it is known that $v'$ will equal

$$\sqrt{\frac{2f}{r} - \frac{f}{a}}. \tag{4}$$

Substituting this value of $v'$ in (3), and omitting the constant $f$, we have to minimize absolutely the expression

$$U = \int_{v_0}^{v_1} \sqrt{\frac{2}{r} - \frac{1}{a}}\, \sqrt{r^2 + r'^2}\, dv$$

$$= \int_{v_0}^{v_1} W \sqrt{r^2 + r'^2}\, dv = \int_{v_0}^{v_1} V dv. \tag{5}$$

Now change $r$ into $r + \delta r$, and $r'$ into $r' + \delta r'$, while $v$ remains unvaried. Then we can develop the new state of $U$ just as we could if in $U$ we had put $x$ for $v$, $y$ for $r$, and $y'$ for $r'$. Hence, to the first order, we have

$$\delta U = \int_{v_0}^{v_1} \left\{ \left( \frac{Wr}{\sqrt{r^2 + r'^2}} - \frac{\sqrt{r^2 + r'^2}}{Wr^2} \right) \delta r + \frac{Wr'}{\sqrt{r^2 + r'^2}}\, \delta r' \right\} dv. \tag{6}$$

But, as in plane co-ordinates, $\delta r' = \frac{d\delta r}{dv}$, so that $\delta U$ may be transformed in the usual manner by integration by parts, $\delta r_1$ and $\delta r_0$ vanishing because the two radii are fixed. But we need not perform this transformation, which would give an expression not readily integrable. For the formulæ of Art. 56 become at once applicable to polar co-ordinates when in those formulæ we substitute $v$, $r$, $r'$, $r''$, etc., for $x$, $y$, $y'$, $y''$, etc. Here, then, $V$ is a function of $r$ and $r'$, and

$$P \quad \text{or} \quad \frac{dV}{dr'} = \frac{Wr'}{\sqrt{r^2 + r'^2}}, \tag{7}$$

so that by formula (C), Art. 56, we have

$$W\sqrt{r^2 + r'^2} = \frac{Wr'^2}{\sqrt{r^2 + r'^2}} + c, \qquad \text{and} \qquad \frac{Wr^2}{\sqrt{r^2 + r'^2}} = c. \tag{8}$$

Solving for $r'^2$, we obtain

$$r'^2 = \frac{W^2 r^4}{b} - r^2, \tag{9}$$

where $b = c^2$. Now put $\frac{1}{u}$ for $r$. Then the following equations will be found to hold true:

$$W^2 = 2u - \frac{1}{a}, \qquad r'^2 = \frac{1}{u^4}\frac{du^2}{dv^2};$$

and (9) gives

$$\frac{du^2}{dv^2} = \frac{2u}{b} - \frac{1}{ab} - u^2. \tag{10}$$

Solving and putting $C$ for $\frac{1}{ab}$, we have

$$dv = -\frac{du}{\sqrt{\frac{2u}{b} - u^2 - C}}, \tag{11}$$

where the negative sign is used, because $\frac{du}{dv} = -\frac{1}{r^2}\frac{dr}{dv}$. Now by placing $\frac{1}{b^2} - \frac{1}{b^2}$ within the radical sign in (11), that equation may evidently be written thus:

$$dv = \frac{-du}{\sqrt{\left(\frac{1}{b^2} - C\right) - \left(u - \frac{1}{b}\right)^2}} = \frac{dX}{\sqrt{R^2 - X^2}}. \tag{12}$$

Whence, by integration, we obtain

$$v + g = \cos^{-1}\frac{X}{R} = \cos^{-1}\frac{u - \frac{1}{b}}{\sqrt{\frac{1}{b^2} - C}}.$$

Whence

$$\cos(v + g) = \frac{u - \frac{1}{b}}{\sqrt{\frac{1}{b^2} - C}}, \tag{13}$$

and

$$u \quad \text{or} \quad \frac{1}{r} = \frac{1}{b} + \sqrt{\frac{1}{b^2} - C}\cos(v + g). \tag{14}$$

Now write $b = a(1 - e^2)$, and replace $C$ by its value $\frac{1}{ab}$. Then the quantity under the radical readily reduces to $\frac{e}{a(1 - e^2)}$, and we have

$$\frac{1}{r} = \frac{1 + e\cos(v + g)}{a(1 - e^2)}. \tag{15}$$

Now in equation (8), in order that $c$, or $\sqrt{b}$, may be a real quantity, we must, since $a$ is by supposition positive, have

$1 - e^2$ positive. That is, $e$ must be less than unity, and (15) is therefore the equation of an ellipse.

**122.** It appears as though the general solution contained four arbitrary constants; but as $e$ depends upon the ratio of $a$ and $b$, the semi-major and minor axes, the number of arbitrary constants is only three. But, as in former examples, the general solution is totally independent of the possibility of rendering it applicable in any particular case. Of these constants, $a$, or the semi-major axis, is determined as soon as $f$, $r_0$ and $v_0'$ are given, but must of course be of sufficient value to enable the ellipse to pass through the second fixed point. The least value of $a$ which will render the solution possible in any case may be determined thus: Since the distance of the two fixed points from the first focus are respectively $r_0$ and $r_1$, their respective distances from the second focus must be $2a - r_0$ and $2a - r_1$. Now from the first fixed point, with a radius $2a - r_0$, and from the second, with a radius $2a - r_1$, describe circular arcs. Then if these arcs do not touch there can be no solution, the least admissible value of $a$ being that which will cause them to touch, while if $a$ be increased beyond this value, the circles will cut, and there will be two positions for the second focus, that is, two ellipses can be described as required.

Thus, although we seem to have three conditions for the determination of the three constants—namely, the intensity of the initial velocity and the distance of each of the two fixed points from the focus—we can in fact only determine $a$. This result might, however, have been anticipated, as we know from mechanics that while the form of the curve and the value of its major axis depend solely upon the values of $f$, $v_0'$ and $r_0$, the minor axis, $2b$, is also dependent upon the direction of the initial velocity, the equation of condition being

$$b = v_0' r_0 \sqrt{\frac{a}{f}} \sin m_0, \tag{16}$$

where $m_0$ is the angle which the orbit at the point $r_0$ makes with $r_0$; and this element of direction we have thus far entirely ignored. If now we assign the value of $m_0$, $b$ and consequently $e$ will be given by (16), and $g$ must then be determined by making the ellipse pass through the two fixed points.

When $a$ has its least value, so that but one ellipse can be described, the chord joining the two fixed points is evidently a focal chord; and when $a$ permits two ellipses to be described, one of them will have its foci upon opposite sides of this chord, while the other will have both upon the same side. This distinction is important, as we shall subsequently show by Jacobi's method that only when the ellipse is of the latter species does it render the action a minimum.

**123.** If, with a fixed value of $r_0$ and $v_0'$, we regard $m_0$ as variable, and for each value of $m_0$ cause the second fixed point $B$ to assume the corresponding position, which would render one solution only possible, the point $B$ will itself always be found upon the perimeter of an ellipse. For there being but one solution, if $D$ be the chord joining the two fixed points, the circles described as above will just touch on $D$, and we shall have

$$2a - r_0 + 2a - r_1 = D, \qquad \text{or} \quad D + r_1 = 4a - r_0.$$

But $D$ and $r_1$ are variable, while $a$ and $r_0$ are constant. Therefore, since the point $B$ is always so situated that the sum of its distances from the first fixed point and the centre of force is always equal to a constant, it is on an ellipse whose foci are at these two points, whose major axis is $4a - r_0$, and whose eccentricity is $\frac{r_1}{4a - r_0}$; and we may call this ellipse the limiting ellipse.

**124.** We may, in closing, advert to the two remaining cases of this problem.

Suppose, first, that we make $v_0'$ equal to $\sqrt{\frac{2f}{r_0}}$. Then it is known that $v'$ will equal $\sqrt{\frac{2f}{r}}$; and proceeding precisely as in the former case, or better by making $C$ zero in equation (14), (since that equation is true when $\frac{1}{a}$ is zero,) we shall obtain

$$\frac{1}{r}=\frac{1+e\cos(v+g)}{b}, \tag{17}$$

the equation of a parabola, in which $b$ is one half the latus rectum.

Suppose, secondly, that we have $v_0'=\sqrt{\frac{2f}{r_0}+\frac{f}{a}}$. Then we know that $v'$ will always equal $\sqrt{\frac{2f}{r}+\frac{f}{a}}$; and proceeding in all respects as before, we shall obtain, in the place of equation (14),

$$\frac{1}{r}=\frac{1}{b}+\sqrt{\frac{1}{b^2}+C}\cos(v+g), \tag{18}$$

where $C$ has the same value as in (14). If now we write $b=-a(1-e^2)$, (18) will readily reduce to

$$\frac{1}{r}=-\frac{1\pm c\cos(v+g)}{a(1-e^2)}. \tag{19}$$

But we shall, in the course of the investigation, obtain an equation identical in form with (8), except that $W$ will equal $\sqrt{\frac{2}{r}+\frac{1}{a}}$. Hence, that $c$ or $\sqrt{b}$ may be real, $b$ or $-(1-e^2)$ must be positive; and therefore, since $a$ is by supposition positive, it readily appears that $1-e^2$ is negative; so that

since $e$ in this case is greater than unity, (19) becomes the equation of an hyperbola, having its attracting focus within the curve. This is as it should be, since a particle, revolving in an orbit according to the Newtonian law, can never describe an hyperbolic arc having the attracting focus without the curve.

## Problem XXIII.

**125.** *It is required to determine the form of the plane closed curve of given length which will envelop a maximum area.*

Assume the pole within the figure, and let $l$ be the length of the given perimeter. Then, because the curve is to be closed, we have

$$l = \int_0^{2\pi} \sqrt{r^2 + r'^2}\, dv, \tag{1}$$

which is to remain constant. Now $m$ being the enclosed area, we have, by the principle of polar areas, $dm = \frac{r^2}{2} dv$, so that we have

$$m = \int_0^{2\pi} \frac{r^2 dv}{2}, \tag{2}$$

which must become a maximum.

Now the reasoning of Bertrand, in Arts. 92 and 93, is evidently rendered applicable to polar co-ordinates by substituting $v$, $r$, $r'$, etc., for $x$, $y$, $y'$, etc. Whence we conclude that Euler's method may be used for polar co-ordinates just as it has been hitherto employed. We must, then, maximize absolutely the expression

$$U = \int_0^{2\pi} \left\{ \frac{r^2}{2} + a\sqrt{r^2 + r'^2} \right\} dv = \int_0^{2\pi} V dv. \tag{3}$$

Here $V$ is a function of $r$ and $r'$, and

$$P = \frac{ar'}{\sqrt{r^2 + r'^2}}, \tag{4}$$

so that by formula (C), Art. 56, we have

$$\frac{r^2}{2} + a\sqrt{r^2 + r'^2} = \frac{ar'^2}{\sqrt{r^2 + r'^2}} + c,$$

and

$$r^2 + \frac{2ar^2}{\sqrt{r^2 + r'^2}} = 2c. \tag{5}$$

Therefore

$$-\frac{2ar^2}{\sqrt{r^2 + r'^2}} = r^2 - 2c. \tag{6}$$

Whence

$$r'^2 = \frac{4a^2r^4}{(r^2 - 2c)^2} - r^2 = r^2\frac{4a^2r^2 - (r^2 - 2c)^2}{(r^2 - 2c)^2}. \tag{7}$$

Hence

$$\frac{dv}{dr} = \frac{r^2 - 2c}{r\sqrt{4a^2r^2 - (r^2 - 2c)^2}}. \tag{8}$$

Now squaring $r^2 - 2c$ under the radical sign, dividing both numerator and denominator by $r^2$, and then placing within the radical the quantity $4c - 4c$, (8) may be written thus:

$$dv = \frac{\left(1 - \frac{2c}{r^2}\right)dr}{\sqrt{4a^2 + 8c - \left(r + \frac{2c}{r}\right)^2}}. \tag{9}$$

Write

$$Z = r + \frac{2c}{r}. \tag{10}$$

Then (9) becomes

$$dv = \frac{dZ}{\sqrt{4a^2 + 8c - Z^2}}. \tag{11}$$

Therefore, by integration, we obtain

$$v + g = \sin^{-1}\frac{Z}{\sqrt{4a^2 + 8c}}, \tag{12}$$

and

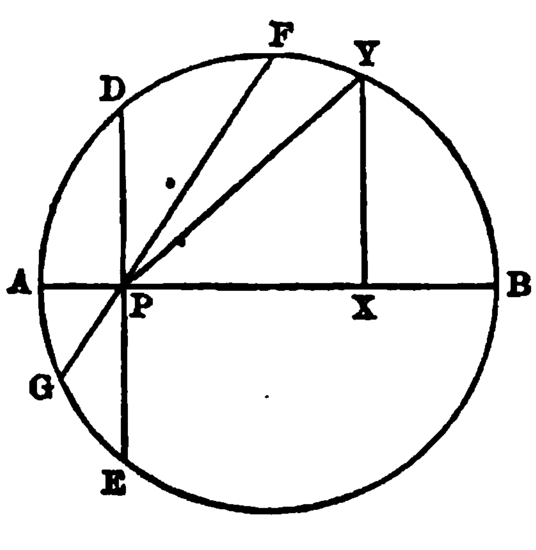
$$\sin(v + g) = \frac{Z}{\sqrt{4a^2 + 8c}}. \tag{13}$$

Clearing fractions and restoring the value of $Z$, then clearing fractions again and transposing the first member, we obtain

$$r^2 - 2r\sqrt{a^2 + 2c}\sin(v + g) + 2c = 0, \tag{14}$$

which is one form of the polar equation of the circle when the pole is assumed at pleasure, $a$ being the radius.

**126.** Equation (14) is the form in which the result is left by Prof. Todhunter. (See his History of Variations, Art. 99.) To interpret this result, let $P$ be the pole, $APB$ a diameter, and denote $PA$ by $C$.




Then since the equation of the circle, when the origin is at $A$, $a$ being its radius, is $y^2 = 2ax - x^2$, if we remove the origin to $P$, it will become

$$y^2 = 2a(x + C) - (x + C)^2. \tag{15}$$

Now, in passing to polar co-ordinates, let $r = PY$ be the radius vector, and $AB$ the initial line. Then we have $x = r\cos v$, and $y = r\sin v$. Substituting these values in (15), and performing the indicated squaring, we easily obtain by transposing, observing that $\sin^2 v + \cos^2 v = 1$,

$$r^2 = 2aC - C^2 + 2r(a - C)\cos v$$

$$= 2aC - C^2 + 2r\sqrt{a^2 - 2aC + C^2}\cos v. \qquad (16)$$

Now put $2c$ for $-2aC + C^2$, and also put for $\cos v$ the sine of its complement, $v'$. Then transposing the second member of (16), and putting $v$ for $v'$, or the angle $DPY$, it becomes

$$r^2 - 2r\sqrt{a^2 + 2c}\sin v + 2c = 0; \qquad (17)$$

and by assuming any other initial, as $FG$, it is plain that the present $v$ will become $v$ plus some constant, say $g$.

**127.** In this problem the terms at the limits, which should be

$$V_1 dv_1 - V_0 dv_0 + P_1\delta r_1 - P_0\delta r_0,$$

present a marked peculiarity. For, since the curve is to be closed, we must consider the limits of integration, viz., 0 and $2\pi$, to be fixed, so that the terms become merely $P_1\delta r_1 - P_0\delta r_0$. Moreover, $r_0$ and $r_1$ become one and the same radius vector, and the terms at the limits therefore vanish without causing $\delta r_1$, $\delta r_0$, $P_1$, or $P_0$ to vanish. Hence these terms furnish no conditions for the determination of the arbitrary constants which enter the general solution. These constants, therefore, with the exception of $a$, which is fixed when the length of the curve is assigned, must remain undetermined. But this should not be otherwise. For we see from the last article that $g$ is numerically equal to the angle $YPF$, while $c$ depends upon the position of the pole with relation to the centre; and we can

evidently, without affecting the result, assume any pole and any initial line we please.

If, however, we had required that a curve of given length should pass through two fixed points, and should, together with the radii to these points, include a maximum area, the three constants would be determined from the assigned length of the arc, combined with the two equations which would hold in order that it might pass through the two given points.

In leaving this subject, we may remark that whatever has been shown concerning the general treatment of the limiting terms in problems of rectangular co-ordinates will be equally applicable here. Thus, if the limiting values of $v$ only be assigned, while those of $r$, $r'$, etc., are subject to variation, we must equate the coefficients of $\delta r_1$, $\delta r_1'$, $\delta r_0$, $\delta r_0'$, etc., severally to zero. If it become necessary to vary the limiting values of $v$ also, we change $v_1$ into $v_1 + dv_1$, and $v_0$ into $v_0 + dv_0$; and if the required curve is to have its extremities upon two other curves, equations (10) of Art. 69, or the more simple equations (2) of Art. 76, will be applicable when we put $v$ for $x$, $r$ for $y$, $r'$ for $y'$, etc.

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## SECTION VIII.

### *DISCRIMINATION OF MAXIMA AND MINIMA (JACOBI'S THEOREM).*

**128.** We have already seen that, in discussing the maximum or minimum state of any definite integral, we must equate the terms of the first order in its variation to zero, and then, having solved the differential equation obtained thereby, this solution must, if it do not reduce the terms of the second order to zero also, render them positive for a minimum and negative for a maximum. We have also seen that the method

of transforming these terms, so as to render their sign evident, has been far from uniform, while in some cases we have been unable to investigate the sign of these terms at all. We now proceed to explain Jacobi's Theorem, which gives us an invariable method of investigating the sign of these terms when the limiting values of $x, y, y'$, etc., are fixed. But as the general discussion is somewhat abstruse, we shall begin with the most simple case, which is also the one which will most frequently present itself for consideration.

## Case 1.

Assume the equation

$$U = \int_{x_0}^{x_1} V dx, \tag{1}$$

where $V$ is any function of $x$, $y$ and $y'$ only. Then to the second order, inclusive, we have

$$\delta U = \int_{x_0}^{x_1} \left\{ \frac{dV}{dy} \delta y + \frac{dV}{dy'} \delta y' \right\} dx$$

$$+ \frac{1}{2} \int_{x_0}^{x_1} \left\{ \frac{d^2 V}{dy^2} \delta y^2 + \frac{2d^2 V}{dy\,dy'} \delta y\,\delta y' + \frac{d^2 V}{dy'^2} \delta y'^2 \right\} dx, \tag{2}$$

the limiting values of $x$ being fixed. Now the terms of the first order, when transformed in the usual manner, become

$$P_1 \delta y_1 - P_0 \delta y_0 + \int_{x_0}^{x_1} M \delta y\, dx,$$

where

$$P = \frac{dV}{dy'}, \qquad M = N - \frac{dP}{dx} = \frac{dV}{dy} - \frac{d}{dx}\frac{dV}{dy'}.$$

But if we would render $U$ a maximum or minimum, the solution of our problem must be the value of $y$ obtained by

completely integrating the equation $M = 0$; and since this is an equation of the second order, this value of $y$ will certainly be some function of $x$ and two arbitrary constants, say

$$y = f(x, c_1, c_2) = f. \tag{3}$$

Of course other constants may enter $V$, and consequently $y$, but with these we are not now concerned. Then, since the form of the function $f$ will be independent of the conditions which are to hold at the limits, we must next so determine $c_1$ and $c_2$ as to satisfy these conditions, and then the solution becomes complete so far as the terms of the first order are concerned.

**129.** The foregoing considerations will prepare us for the discussion of the terms of the second order; but before entering upon the explanation of Jacobi's Theorem, we may say that its object in the present case is to put the terms of the second order under the form $\frac{d^2V}{dy'^2}$ multiplied by the square of a certain function, and also to determine the form of this function.

Now, since the terms of the first order must vanish, there remain only terms of the second and higher orders, and we may, to the second order, write

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}(a\delta y^2 + 2b\delta y\,\delta y' + c\delta y'^2)\,dx, \tag{4}$$

where $a$, $b$ and $c$ have the values shown in (2).

Let as assume that $\delta y_1$, $\delta y_0$ are zero; then we shall first show that $\delta U$ can be written thus:

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\left\{A\delta y + \frac{d}{dx}A_1\delta y'\right\}\delta y\,dx, \tag{5}$$

where $A$ and $A_1$ are variable functions, the suffix 1 having

no reference to limits. Observing that $\delta y' = \dfrac{d\delta y}{dx}$, we have, by parts,

$$\int c\delta y'^2 dx = c\delta y'\delta y - \int \delta y \frac{d}{dx} c\delta y'.dx. \tag{8}$$

Also

$$\int b\delta y\,\delta y' dx = b\delta y^2 - \int \delta y \frac{d}{dx} b\delta y.dx$$

$$= b\delta y^2 - \int b\delta y\,\delta y' dx - \int \frac{db}{dx}\delta y^2 dx. \tag{9}$$

Hence

$$2\int b\delta y\,\delta y' dx = b\delta y^2 - \int \frac{db}{dx}\delta y^2 dx. \tag{10}$$

Therefore, collecting results, arranging and factoring, we have

$$\delta U = \frac{1}{2}\left\{ (b\delta y^2)_1 - (b\delta y^2)_0 + (c\delta y\,\delta y')_1 - (c\delta y\,\delta y')_0 \right\}$$

$$+ \frac{1}{2}\int_{x_0}^{x_1} \left\{ \left(a - \frac{db}{dx}\right)\delta y - \frac{d}{dx} c\delta y' \right\} \delta y\, dx, \tag{11}$$

which, when we make $\delta y_1$ and $\delta y_0$ vanish, gives $\delta U$ in the required form, and

$$A = a - \frac{db}{dx}, \qquad A_1 = -c.$$

**130.** We will now show, in the second place, that if we vary $M$, we may also write

$$\delta M = A\delta y + \frac{d}{dx} A_1 \delta y'. \tag{12}$$

We have

$$M = \frac{dV}{dy} - \frac{d}{dx}\frac{dV}{dy'} = N - \frac{dP}{dx}.$$

Varying the first term, we have $a\delta y + b\delta y'$; and varying $P$, we obtain $d\delta y + c\delta y'$. Hence the variation of $-\frac{dP}{dx}$ (that is, the change which it undergoes from changing $y$ into $y + \delta y$, and $y'$ into $y' + \delta y'$,) is $-\frac{d}{dx}(b\delta y + c\delta y')$, or, by differentiation,

$$-b\delta y' - \frac{db}{dx}\delta y - \frac{d}{dx}c\delta y'.$$

Collecting and arranging, we have

$$\delta M = \left(a - \frac{db}{dx}\right)\delta y - \frac{d}{dx}c\delta y' = A\delta y + \frac{d}{dx}A_1\delta y', \tag{13}$$

and therefore we may, if $\delta y_1$ and $\delta y_0$ vanish, write

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\delta M\,\delta y\,dx. \tag{14}$$

**131.** We have already shown that if the terms of the second order in $\delta U$ vanish, we shall be obliged to examine those of the third; and as these will not usually vanish, but will be positive or negative at our pleasure, we shall be, in general, safe in assuming that in this case we have neither a maximum nor a minimum state of $U$. But it is evident that the quantities $A$ and $A_1$ are not at all in our power, so that unless those quantities vanish of themselves the terms of the second order can only be made to disappear by the assumption of suitable values of $\delta y$ and $\delta y'$.

Now let $u$ be such a quantity as will satisfy the equation

$$Au + \frac{d}{dx}A_1u' = 0, \tag{15}$$

where $u' = \frac{du}{dx}$. Then it is clear that if $\delta y$ throughout the definite integral can be taken equal to $u$, or to $ku$, where $k$ is any constant, $\delta U$ to the second order will vanish. Of course since $\delta y$ and $\delta y'$ must be infinitesimal, $k$ must be also infinitesimal, unless $u$ be already so.

**132.** We will next determine the quantity $u$, as we shall then be better able to see how it may be employed. We have seen that the value of $y$ obtained by the complete integration of the equation $M = 0$ will be of the form $y = f(x, c_1, c_2) = f$, and that this value of $y$ will satisfy the above differential equation independently of the value of $c_1$ and $c_2$. If, therefore, we make any changes in the form of the values of these constants, the resulting changes in $y$ and its differential coefficients, while not necessarily zero, will not prevent these quantities from still causing $M$ to vanish. Now suppose we change $c_1$ into $c_1 + \delta c_1$, and $c_2$ into $c_2 + \delta c_2$, where $\delta c_1$ and $\delta c_2$ are infinitesimal but independent constants. Then denoting by $\delta' y$ and $\delta' y'$ the corresponding changes in $y$ and $y'$, we shall have

$$\delta' y = \frac{df}{dc_1}\delta c_1 + \frac{df}{dc_2}\delta c_2 \tag{16}$$

and

$$\delta' y' = \frac{d}{dx}\left(\frac{df}{dc_1}\delta c_1 + \frac{df}{dc_2}\delta c_2\right). \tag{17}$$

Hence these values of $\delta y$ and $\delta y'$, if admissible throughout the range of integration, will render the corresponding variation, $\delta' M$, zero throughout those limits, and will also, as we see from (14), render $\delta' U$ zero. But we shall find it convenient to write

$$\delta' y = k\left(\frac{df}{dc_1} + l\,\frac{df}{dc_2}\right), \tag{18}$$

where $k = \delta c_1$ and $l = \frac{\delta c_2}{\delta c_1}$; and as $\delta c_1$ and $\delta c_2$ are entirely independent, we can make $l$ assume any real and constant value we please.

We conclude, then, from (13) and (15), that the general valuc of $u$, if not infinitesimal, is

$$u = \frac{df}{dc_1} + l\,\frac{df}{dc_2}. \tag{19}$$

But although this is the most general form of $u$, it is evident that we need not vary both constants in $f$, so that we may have

$$ku = \frac{df}{dc_1}\,\delta c_1 \qquad \text{or} \qquad ku = \frac{df}{dc_2}\,\delta c_2. \tag{20}$$

**133.** Let us next consider whether $ku$ can be an admissible value of $\delta y$ throughout $U$; because if it can, there will be no need of any further transformation of the terms of the second order, since there will be at least one mode of varying $y$ which will cause these terms to vanish.

We observe, first, that since $\delta y$ and $\delta y'$ must be always infinitesimal, if $ku$ be an admissible variation of $y$ for any portion of the integral, say from $x_2$ to $x_3$, $u$ and $u'$ must remain finite throughout these limits.

In the second place, if $ku$ be an admissible variation of $y$ throughout a portion only of the required curve, say from $x_2$ to $x_3$, while the values of $x_0, y_0, x_1, y_1$ are fixed, then to certainly make the terms of the second order vanish we must have $y_2$ and $y_3$ also fixed; must change $y$ into $y + ku$ throughout the limits $x_2$ and $x_3$, and leave the rest of the required curve unvaried. As this requires that $u$ shall vanish, both when $x = x_2$ and when $x = x_3$, and as $\delta y$ could not equal $ku$ throughout any limits unless $u$ vanish at both those limits, we conclude generally that to make the terms of the second order

disappear by the use of $ku$ for $\delta y$, $u$ must vanish at least twice within the limits of integration.

In the third place, if either of the quantities $\frac{df}{dc_1}$ or $\frac{df}{dc_2}$, which are not in our power, vanish twice within the range of integration, while at the same time its first differential coefficient with respect to $x$ remains always finite, we can make the terms of the second order disappear by putting that quantity for $u$, but not otherwise.

Moreover, that we may employ the general value of $ku$, all the quantities $\frac{df}{dc_1}$, $\frac{d}{dx}\frac{df}{dc_1}$, $\frac{df}{dc_2}$ and $\frac{d}{dx}\frac{df}{dc_2}$, must remain finite throughout the limits for which $ku$ is employed, and we must also be able to so assume $u$ that it shall vanish at least twice as we pass from $x_0$ to $x_1$.

We will now consider under what circumstances this latter condition can be fulfilled. Put $h$ for $\frac{\frac{df}{dc_1}}{\frac{df}{dc_2}}$. Then we see from (19) that we can cause $u$ to vanish for any value of $x$ we please, say for $x = x_2$, by taking $l = -h_2$; and this is all that we can effect. We can, moreover, in some cases assume $u$ so that it shall not vanish as we pass from $x_0$ to $x_1$, while in other cases we cannot. For our power over $u$ depends entirely upon our assumption of $l$. Now suppose we find that $h$, which is not in our power, cannot assume all possible values from negative to positive infinity as we pass from $x_0$ to $x_1$. Then, by assuming $l$ equal to one of these values, but multiplied by $-1$, we can effect that $u$ shall not vanish within the limits $x_0$ and $x_1$. But if, on the other hand, we find that $h$ ranges through all real values, we cannot assume $l$ so that $u$ shall not vanish at least once.

To apply the foregoing, assume $l$ so that $u$ shall vanish when $x = x_0$. Then if the range of $h$ through all real values

be complete, $u$ will evidently vanish again at or before the upper limit, according as $h$ may complete or more than complete its range, and we can make the terms of the second order vanish by the use of $ku$. But if the range of $h$ be only partial, $u$ will not vanish again at or before the upper limit, and we cannot employ $ku$ to make those terms disappear.

**134.** It is evident that when $ku$ cannot be employed to make the terms of the second order vanish, some further transformation will be necessary to render their sign apparent; and to this we now proceed.

Let $u$ involve $k$—that is, be $ku$—so that it may be infinitesimal, and resume the equations

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \left\{ A\delta y + \frac{d}{dx} A_1 \delta y' \right\} \delta y \, dx \tag{21}$$

and

$$Au + \frac{d}{dx} A_1 u' = 0. \tag{22}$$

Then whatever be the value of $\delta y$, we may certainly make it equal to $ut$, and (21) will then become

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \left\{ Aut + \frac{d}{dx} A_1 (ut)' \right\} ut \, dx, \tag{23}$$

where $(ut)' = \frac{d}{dx} ut$.

We wish now to reduce (23) by integrating it by parts; but before doing so we must show that because (22) is true, the expression

$$u \left\{ Aut + \frac{d}{dx} A_1 (ut)' \right\} dx \quad \text{or} \quad W dx \tag{24}$$

can always be integrated, its integral taking the form $B_1 t'$, where $B_1$ is a new variable function, the suffix 1 having no ref- reference to limits, and $t' = \frac{dt}{dx}$.

**135.** Multiply (22) by $ut$, and subtract the product from the value of $W$ in (24), and we have

$$W = u \left\{ \frac{d}{dx} A_1 (ut)' - ut \frac{d}{dx} A_1 u' \right\}. \tag{25}$$

Now

$$u \frac{d}{dx} A_1 (ut)' = \frac{d}{dx} u A_1 (ut)' - A_1 (ut)' u'. \tag{26}$$

But $(ut)' = ut' + tu'$. Whence

$$u \frac{d}{dx} A_1 (ut)' = \frac{d}{dx} u^2 A_1 t' + \frac{d}{dx} u A_1 t u' - A_1 u' u t' - A_1 u'^2 t \tag{27}$$

and

$$\frac{d}{dx} u A_1 u' t = u A_1 u' t' + t \frac{d}{dx} u A_1 u'.$$

Whence

$$u \frac{d}{dx} A_1 (ut)' = \left\{ \frac{d}{dx} u A_1 u' - A_1 u'^2 \right\} t + \frac{d}{dx} u^2 A_1 t'. \tag{28}$$

Now if the differentiation indicated in the first member of (28) were performed, it is evident that the only term in which $t$ could appear undifferentiated would be

$$ut \frac{d}{dx} A_1 u' \qquad \text{or} \qquad \left\{ \frac{d}{dx} u A_1 u' - A_1 u'^2 \right\} t.$$

Hence we see from (25) that the terms in $W$ which contain $t$ will cancel, and we shall have

$$W = \frac{d}{dx} u^2 A_1 t' = \frac{d}{dx} B_1 t',$$

where

$$B_1 = u^2 A_1 \tag{29}$$

and

$$\int W dx = \int \frac{d}{dx} B_1 t' . dx = B_1 t', \tag{30}$$

the constant being neglected.

**136.** By the use of (30), (23) may now be integrated by parts thus:

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} Wt\,dx \tag{31}$$

$$= \frac{1}{2}\left\{ (tB_1t')_1 - (tB_1t')_0 \right\} - \frac{1}{2}\int_{x_0}^{x_1} B_1 t'^2 dx.$$

Now examining equations (29), (11), (4) and (2), we see that

$$B_1 = u^2 A_1 = -u^2 c = -\frac{d^2 V}{dy'^2} u^2; \tag{32}$$

and since we put $\delta y$ equal to $ut$, we have

$$t' = \frac{u\delta y' - \delta y u'}{u^2}. \tag{33}$$

If the terms without the integral sign in (31) do not vanish, they must be added to those already in (11). But the supposition that $\delta y_1$ and $\delta y_0$ are zero will certainly reduce these terms to zero unless $u_1$ and $u_0$ vanish, which would, as we have seen, indicate generally that there is neither a maximum nor a minimum. Therefore, finally substituting for $B_1$ and $t'$ their values from (32) and (33), we have

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \frac{d^2 V}{dy'^2} \frac{(u\delta y' - \delta y u')^2}{u^2} dx$$

$$= \frac{1}{2}\int_{x_0}^{x_1} \frac{d^2 V}{dy'^2} \frac{(u'\delta y - u\delta y')^2}{u^2} dx; \tag{34}$$

and if we now consider $u$ as no longer involving $k$, we must multiply the last member by $k^2$.

**137.** Let us now consider the last equation more particularly

*First.* We shall assume that before obtaining this equation it had been ascertained that the terms of the second order could not be reduced to zero by any use of $ku$ for $\delta y$; that is, that $u$ could be so assumed as not to vanish at all, since otherwise the last transformation would be needless.

*Second.* Now suppose the second factor of (24) does not vanish permanently, in which case it will evidently be positive; and also that it remains finite throughout the range of integration. Then for a maximum or a minimum we require only that $\frac{d^2V}{dy'^2}$ or $c$ shall remain finite, shall not vanish permanently, and shall be of invariable sign. For we have already seen that infinite values cause the method of development employed to become inapplicable, and even in the case of a single element of an integral, render the entire result doubtful. Moreover, if $c$ can change its sign, we can, as has been previously shown, vary $y$ for such values of $x$ as will render $c$ negative, while leaving $y$ unvaried for all other values of $x$, and thus make $\delta U$ negative; or by pursuing a similar course with such values of $x$ as render $c$ positive, we can make $\delta U$ positive. But if $c$ remain finite, be of invariable sign, and do not vanish permanently, we shall have a maximum or a minimum according as it is negative or positive.

*Third.* But suppose the second factor of (34) does vanish. Then we must have

$$u'\delta y - u\delta y' = 0. \tag{35}$$

Whence

$$\frac{u'}{u}\,dx = \frac{\delta y'}{\delta y}\,dx, \qquad \text{or} \qquad \frac{du}{u} = \frac{d\delta y}{dy}.$$

Therefore $l\delta y = lu + g = lu + lk = l(ku)$, and $\delta y = ku$, where $k$ is any infinitesimal constant. But by supposition the problem is such that $\delta y$ cannot be made equal to $ku$ throughout the range of integration, and therefore the second factor of (34) will not vanish permanently.

Hence we see that if the terms of the second order cannot be reduced to zero by the use of $ku$, then unless $c$ vanish they cannot be reduced to zero by any admissible mode of varying $y$, and this supplies what was before wanting in the complete investigation of the subject. To render the second factor of (34) infinite, we must, if $\delta y$ and $\delta y'$ be infinitesimal, have either $u = 0$ or $u' = \infty$. But the first condition disappears, since we suppose $u$ to be taken so as not to vanish at all, and the second cannot occur unless $\frac{d}{dx}\frac{df}{dc_1}$ or $\frac{d}{dx}\frac{df}{dc_2}$ become infinite.

It will be seen that the expression $u\delta y' - \delta y u'$ in (34) is the determinant of $u$, $u'$, $\delta y$, $\delta y'$; so that, putting $D$ for their determinant, we may write

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} c \frac{D^2}{u^2}\, dx,$$

and we shall see hereafter that determinants can always be employed in expressing the final results of Jacobi's transformation.

**138.** Before applying this theorem to any example the following general directions may be useful.

First. Having obtained the general solution, find $\frac{d^2V}{dy'^2}$ or $c$, which must not vanish permanently, become infinite, nor change its sign. For in the first case the terms of the second order would reduce to zero; in the second the investigation would become more or less unsatisfactory; while in the third the terms of the second order can be made to assume either sign, thus rendering a maximum or a minimum impossible.

Second. If these conditions be satisfactory, find the quantities $\frac{df}{dc_1}$ and $\frac{df}{dc_2}$, neither of which must vanish twice within the

range of integration, otherwise we can reduce the terms of the second order to zero by employing this quantity for $u$.

Third. Moreover, the first differential coefficients of these quantities with respect to $x$ should remain finite as we pass from $x_0$ to $x_1$, otherwise some element of $\delta U$ may become infinite, thus rendering the result untrustworthy.

Fourth. If all these conditions still indicate a maximum or a minimum, consider next whether, in the general value of $u$, $h$ or the ratio between the quantities $\frac{df}{dc_1}$ and $\frac{df}{dc_2}$ can range over all real values as we pass from $x_0$ to $x_1$. For if it can, the terms of the second order can be made to vanish by the use of $ku$; but if it cannot, those terms cannot be reduced to zero by any admissible values of $\delta y$, and our investigations are complete, assuring us of a maximum or a minimum according as $c$ is negative or positive.

## Problem XXIV.

**139.** *It is required to apply Jacobi's Theorem to Prob. I.*

Here the general solution is

$$y = f(x, c_1, c_2) = f = c_1 x + c_2. \tag{1}$$

Also,

$$V = \sqrt{1 + y'^2},$$

so that

$$\frac{d^2V}{dy'^2} = \frac{1}{\sqrt{(1 + y'^2)^3}}, \tag{2}$$

and this last expression is evidently positive, finite, and of invariable sign. We likewise obtain from (1)

$$\frac{df}{dc_1} = x, \tag{3}$$

and

$$\frac{df}{dc_2} = 1, \tag{4}$$

$$\frac{d}{dx}\frac{df}{dc_1} = 1, \tag{5}$$

and

$$\frac{d}{dx}\frac{df}{dc_2} = 0. \tag{6}$$

Now neither of the first two quantities can vanish twice, nor do their first differential coefficients become infinite. Moreover, if we divide the first of these quantities by the second, we find $h = x$, which will not range through all real values. Hence $u$ can be so assumed as not to vanish at all. For we have $u = x + l$; and by assuming $l$ to be negative and numerically greater than $x_1$, the truth of the assertion becomes evident. Jacobi's Theorem, therefore, indicates a minimum in this case.

### Problem XXV.

**140.** *It is required to apply the theorem of Jacobi to the case of the brachistochrone in Prob. II., Case* 1.

Here, from equation (11), Art. 17, the general solution, which is a cycloid, is seen to be of the form

$$y = f(x, c_1, c_2) = f = c_1 \operatorname{versin}^{-1}\frac{x}{c_1} - \sqrt{2c_1x - x^2} + c_2, \tag{1}$$

where $c_1$ is the radius of the generating circle. We also have

$$V_0 = \frac{\sqrt{1+y'^2}}{\sqrt{x}},$$

so that

$$\frac{d^2V}{dy'^2} = \frac{1}{\sqrt{x(1+y'^2)^3}}. \tag{2}$$

This last expression is of invariable sign and positive, but becomes infinite at the cusp, where both $x$ and $y'$ are zero. The investigation will therefore be subject to any doubt which may arise from this fact. (See closing remark of Art. 21.)

Disregarding this objection, we have from (1), by differentiating carefully with respect to $c_1$ and $c_2$ successively, while treating $x$ as a constant,

$$\frac{df}{dc_1} = \text{versin}^{-1}\frac{x}{c_1} - \frac{2x}{\sqrt{2c_1x - x^2}}, \tag{3}$$

$$\frac{df}{dc_2} = 1. \tag{4}$$

Now we shall take $x_1$ to be somewhat less than $2c_1$. For, as we have seen, $y'$ becomes infinite at the vertex, and we wish as far as possible to avoid infinite quantities, since Jacobi's method does not enable us to overcome the obstacle which these quantities present to a satisfactory solution. With this limitation neither of the above quantities will vanish twice within the range of integration. We also have, by differentiating in the usual way,

$$\frac{d}{dx}\frac{df}{dc_1} = \frac{-\sqrt{x}}{(2c_1 - x)^{\frac{3}{2}}}, \tag{5}$$

$$\frac{d}{dx}\frac{df}{dc_2} = 0, \tag{6}$$

and these quantities remain finite throughout the present limits. Moreover, if we divide $\frac{df}{dc_1}$ by $\frac{df}{dc_2}$, the quotient $h$ will be the second member of (3), and this cannot range over all real values, so that $u$ can be so taken as not to vanish at all as we pass from $x_0$ to $x_1$. We conclude, therefore, that, setting aside the objection previously mentioned, Jacobi's Theorem indicates a minimum in the present case.

## Problem XXVI.

**141.** *It is required to apply the theorem of Jacobi to Prob. XXII.*

From what has been previously said regarding the treatment of polar co-ordinates by the calculus of variations, it will appear that all the reasoning by which Jacobi's transformations were effected will apply also to them when we change $x$ into $v$, $y$ into $r$, and $y'$ into $r'$. We shall consider only the case in which we have an ellipse, our object being to verify the closing remark of Art. 122. We shall, with slight deviations, follow Prof. Todhunter. (See his Researches; or Adams Essay, Art. 183.)

Here, as we see from equation (5), Art. 121,

$$V = \sqrt{\frac{2}{r} - \frac{1}{a}}\,\sqrt{r^2 + r'^2}.$$

Whence

$$\frac{d^2V}{dr'^2} = \frac{\sqrt{\frac{2}{r} - \frac{1}{a}}}{\sqrt{(r^2 + r'^2)^3}},$$

which cannot change its sign, and is always finite and positive. Now the general solution in equation (15), Art. 121, may be written

$$r = f(v, c_1, c_2) = f = \frac{a(1 - e^2)}{1 + e\cos(v + g)}, \tag{1}$$

where $e$ may take the place of $c_1$, and $g$ that of $c_2$.

It appears that (1) contains also another constant, $a$. But this constant was introduced when we assigned the initial velocity, and is not therefore a constant of integration. Now we have already stated that $f$ might involve, besides the independent variable and $c_1$ and $c_2$, any number of other constants;

those only which enter by integration being considered by Jacobi's method.

We must, then, pursue the usual course, and find the differential coefficient of $f$, that is, of $r$ with respect to $e$ and $g$. We have, from (1),

$$\frac{a(1-e^2)}{r} = 1 + e\cos(v+g). \tag{2}$$

Now differentiating with respect to $e$, we obtain

$$-\frac{2ae}{r} - \frac{a(1-e^2)}{r^2}\frac{dr}{de} = \cos(v+g) = \frac{1}{e}\left\{\frac{a(1-e^2)}{r} - 1\right\}, \tag{3}$$

the last member being found from (2). Solving (3), we finally obtain

$$\frac{dr}{de} = \frac{r^2 - ar(1+e^2)}{ae(1-e^2)} \tag{4}$$

and

$$\frac{d}{dv}\frac{dr}{de} = \frac{[2r - a(1+e^2)]r'}{ae(1-e^2)}. \tag{5}$$

Also,

$$\frac{dr}{dg} = \frac{dr}{dv} = r', \tag{6}$$

$$\frac{d}{dv}\frac{dr}{dg} = r''. \tag{7}$$

Now neither the first member of (5) nor (7) can become infinite, so that we may employ Jacobi's Theorem with confidence.

But before resorting to the most general method, let us determine whether the first member of (4) or (6) can vanish twice. Now to make $\frac{dr}{de}$ vanish, we must have

$$r = a(1+e^2). \tag{8}$$

But this is the value of the radius vector drawn to the extremity of the remote latus rectum. For the distance between the foci being $2ae$, and the semi-latus rectum being $a(1-e^2)$, we have

$$r^2 = 4a^2e^2 + a^2(1-e^2)^2 = a^2(1+e^2)^2.$$

Also $r'$, and consequently $\frac{dr}{dg}$, vanishes at each vertex of the ellipse, so that we conclude at once that there will be no minimum if the arc extend from vertex to vertex, or be cut off by the remote latus rectum.

Now, in applying the general method, we are only concerned in knowing the range of $h$, or the ratio of $\frac{df}{dc_1}$ to $\frac{df}{dc_2}$. But $h$ evidently varies as

$$\frac{r^2 - ar(1+e^2)}{r'} \quad \text{or} \quad \frac{1 - \frac{a}{r}(1+e^2)}{\frac{r'}{r^2}}. \tag{9}$$

But

$$\frac{1}{r} = \frac{1 + e\cos(v+g)}{a(1-e^2)}.$$

Whence

$$\frac{r'}{r^2} = \frac{e\sin(v+g)}{a(1-e^2)},$$

and therefore the last member of (9) may be written

$$\frac{1 - \frac{a}{r}(1+e^2)}{\frac{e\sin(v+g)}{a(1-e^2)}}. \tag{10}$$

Now this expression varies only as

$$\frac{1 - \frac{a}{r}(1 + e^2)}{\sin(v + g)} \quad \text{or} \quad \frac{r - a(1 + e^2)}{r \sin(v + g)}. \tag{11}$$

Next let us write

$$r = 2a - R, \tag{12}$$

$$r \sin(v + g) = R \sin w. \tag{13}$$

Then $R$ will be the radius vector drawn from the other focus, and $w$ will become the angle which $R$ makes with the major axis. Then, by substitution, (11) will become

$$\frac{a(1 - e^2) - R}{R \sin w} = \frac{1}{\sin w}\left\{\frac{a(1 - e^2)}{R} - 1\right\} = e \cot w, \tag{14}$$

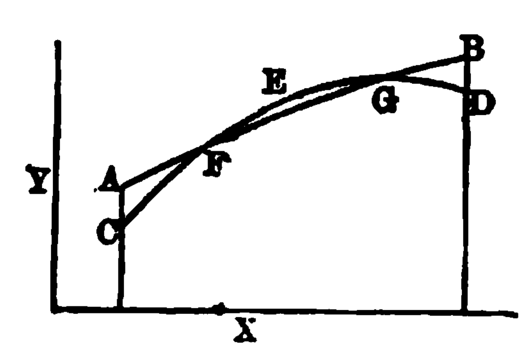
the last member being obtained by substituting for $R$ its value $\frac{a(1 - e^2)}{1 + e \cos w}$, whence $h$ varies as $\cot w$.

Now, in general, any function will have a complete range from negative to positive infinity when we can cause it to start with a given value, change sign by passing through zero or infinity, and return to its initial value. But $\cot w$ becomes infinite at the two vertices only, vanishes only when $r$ is the semi-latus rectum, and changes sign at these four points, and at these only.

Now let $R_0$ and $R_1$ be the radii drawn to the two fixed points. Then, to make $\cot w_0$ and $\cot w_1$ equal, $r_0$ and $r_1$ must form a continuous line; that is, a focal chord. Should the arc extend from one vertex to the other, $\cot w_0$ and $\cot w_1$ will not be equal, but will be infinite and of contrary sign, having passed through zero. But in all other cases $\cot w_0$ and $\cot w_1$ are equal, after having changed sign by passing through infinity.

Here, therefore, there is no minimum, and if the arc be still greater the same remark will hold, unless we were required to vary the entire arc. For since we can make $u$ vanish at each end of the focal chord, we can take $\delta y = ku$ through that portion of the arc, and leave the remainder unvaried, thus making the terms of the second order in $\delta U$ vanish. But if the arc be less than that subtended by a focal chord passing through the present, which is the remote focus—that is, both foci lie upon the same side of the line joining the two fixed points—then the range of cot $w$ will be only partial, and there will be a minimum.

**142.** We may give a general geometrical illustration of Jacobi's method. Let $A$ and $B$ be two fixed points, joined by a curve which satisfies the differential equation $M = 0$, and let $CED$ be another curve derived from the first by such variations of $y$ and $y'$ as will result from varying the constants of integration, and consequently still satisfying the same differential equation.




Then there will, if $\frac{d^2V}{dy'^2}$ permit, be a maximum or a minimum when $CED$ cannot twice meet $AB$ unproduced. But if it can meet it twice, we may regard $AFEGB$ as the new derived curve, which would make the terms of the second order vanish.

But since we can make $u$ vanish once at pleasure, we may suppose the derived curve to touch the other at $A$—that is, we can make $C$ and $A$ coincide—and then we shall have a

maximum or a minimum so long as the other point of meeting, $G$, is not reached.

Moreover, we compare $AB$ with such derived curves only as satisfy the equation $M = 0$, although their number may be infinite. For we have seen that when $ku$ cannot be used to make the terms of the second order disappear, they will not vanish at all if $\frac{d^2V}{dy'^2}$ do not vanish. Hence no other class of curves could render $\delta U$ to the second order zero.

**143.** Now it is evident that, in order to employ the preceding theorem, we must be able to find the functions $\frac{df}{dc_1}$ and $\frac{df}{dc_2}$; that is, to determine the change which $y$ would undergo when in the general solution we give infinitesimal increments to $c_1$ and $c_2$. We therefore naturally first seek to obtain the complete integral of the differential equation $M = 0$, and to exhibit it under the form of $y = f(x, c_1, c_2)$.

But it frequently happens that even when we are unable to obtain the general solution in the explicit form just given, we can still determine the functions $\frac{df}{dc_1}$ and $\frac{df}{dc_2}$. Still this is not strange, since we can often obtain the differential of an unknown quantity; that is, a differential whose integral is unobtainable. When these functions can be found, Jacobi's method can be applied to the investigation of the terms of the second order, whether the equation $M = 0$ can be completely integrated or not; and we now proceed to show how they may be determined in the case of a very important class of problems.

The following method is due to Prof. Todhunter (see his Researches, Arts. 26, 282), and we shall see that by it he has been able to obtain some results not previously known, and to correct some which had been erroneously given.

## Problem XXVII.

**144.** *It is required to discuss in full the conditions which will maximize or minimize the expression*

$$U = \int_{x_0}^{x_1} y^n v \, dx = \int_{x_0}^{x_1} V dx,$$

*where $v$ is any function of $y'$ and constants.*

Here $V$ is a function of $y$ and $y'$ only, and

$$P = \frac{y^n dv}{dy'} = y^n v'.$$

Hence, by formula (C), Art. 56, we have

$$y^n v = y' y^n v' + c_1,$$

whence

$$y^n (v - y'v') = c_1, \tag{1}$$

which is as far as the integration can be carried, so long as $n$ and $v$ are entirely undetermined. But we may suppose a curve to be drawn satisfying (1), and that its equation is $y = f(x, c_1, c_2) = f$. Then, although we cannot determine the form of $f$, we can ascertain what would be the corresponding variation of $y$ if $c_1$ and $c_2$ were increased by $\delta c_1$ and $\delta c_2$, and can then investigate the terms of the second order.

**145.** From (1) we have

$$y = \sqrt[n]{\frac{c_1}{v - y'v'}} = f(y', c_1) = f. \tag{2}$$

Also,

$$dx = \frac{dy}{y'} = \frac{1}{y'} \frac{df}{dy'} dy'.$$

Whence, by supposing the integration performed, we may write

$$x = F(y', c_1) + c_2 = F + c_2. \tag{3}$$

Now, although $f$ and $F$ may contain other constants besides $c_1$, these will not be affected by any variation of $c_1$ or $c_2$, leaving only $y'$ and $c_1$ as variables. Moreover, $x$ will undergo no change when $c_1$ and $c_2$ vary, and these constants themselves are entirely independent of each other. We have then, from (2) and (3),

$$\frac{dy}{dc_1} = \frac{df}{dc_1} + \frac{df}{dy'}\frac{dy'}{dc_1} \tag{4}$$

and

$$0 = \frac{dF}{dc_1} + \frac{dF}{dy'}\frac{dy'}{dc_1}. \tag{5}$$

Whence

$$\frac{dF}{dy'}\frac{dy'}{dc_1} = -\frac{dF}{dc_1}. \tag{6}$$

Differentiating (2) and (3) with respect to $x$, we obtain

$$y' = \frac{y''df}{dy'} \tag{7}$$

and

$$1 = \frac{y''dF}{dy'}. \tag{8}$$

Whence

$$\frac{df}{dy'} = \frac{y'}{y''}. \tag{9}$$

Hence, and then multiplying by (8) and comparing with (6),

$$\frac{df}{dy'}\frac{dy'}{dc_1} = \frac{y'}{y''}\frac{dy'}{dc_1} = \frac{y'dF}{dy'}\frac{dy'}{dc_1} = -\frac{y'dF}{dc_1}. \tag{10}$$

Therefore

$$\frac{dy}{dc_1} = \frac{df}{dc_1} - \frac{y'dF}{dc_1}. \tag{11}$$

Again, from (2) and (3), we have

$$\frac{dy}{dc_2} = \frac{df}{dy'}\frac{dy'}{dc_2} \tag{12}$$

and

$$0 = \frac{dF}{dy'}\frac{dy'}{dc_2} + 1. \tag{13}$$

Comparing this equation with (8), we obtain $\frac{dy'}{dc_2} = -y''$. Whence, by (9),

$$\frac{dy}{dc_2} = -y'. \tag{14}$$

We must next determine the form of $\left(\frac{df}{dc_1}\right)$ and $\left(\frac{dF}{dc_1}\right)$, which are only partial differentials with respect to $c_1$, this fact being indicated by writing them in brackets.

From (2) we have

$$f = c_1^m \frac{1}{(v - y'v')^m}, \tag{15}$$

where $m = \frac{1}{n}$. Hence

$$\left(\frac{df}{dc_1}\right) = \frac{1}{(v - y'v')^m} mc_1^{m-1}. \tag{16}$$

But from (1) we have

$$(v - y'v')^m = \frac{c_1^m}{y^{mn}} = \frac{c_1^m}{y},$$

and therefore, restoring $n$, we have

$$\left(\frac{df}{dc_1}\right) = \frac{y}{nc_1}. \tag{17}$$

Now although we cannot, while $v$ is unknown, determine $F$, still it is evident, from its mode of derivation from $f$, that if $c_1^m$ enter the latter as a factor, it must also enter the former unchanged. $F$ must therefore be of the form $c_1^m w$, where $w$ is some function not involving $c_1$ or $c_2$, but merely $y'$, and perhaps constants, not of integration. Hence, from (3), we have

$$x = c_1^m w + c_2 \tag{18}$$

and

$$w = \frac{x - c_2}{c_1^m}.$$

Now

$$\left(\frac{dF}{dc_1}\right) = wmc_1^{m-1} = \frac{x - c_2}{nc_1}. \tag{19}$$

Therefore, finally,

$$\frac{dy}{dc_1} = \frac{y - y'(x - c_2)}{nc_1}. \tag{20}$$

**146.** Now if the value of $y$ found by the solution of (1) can render $U$ a maximum or a minimum, the terms of the second order in $\delta U$ can be put under the form given in equation (34), Art. 136. Then, supposing $\dfrac{d^2V}{dy'^2}$ or $\dfrac{y^n d^2 v}{dy'^2}$ or $y^n v''$ to be of invariable sign and finite, it will only be necessary that $u$ shall be incapable of vanishing twice; which will in general, as we have seen, follow if it can be so taken as not to vanish at all. Now equations (14) and (20) give us the general value of $u$, thus:

$$u = \frac{dy}{dc_1} + \frac{dy}{dc_2}\frac{\delta c_2}{\delta c_1} = \frac{dy}{dc_1} + l\frac{dy}{dc_2}$$

$$= \frac{1}{nc_1}\left\{y - y'(x - c_2) - Ly'\right\}, \tag{21}$$

where $L = nc_1 l$.

Now by differentiating the last equation with respect to $x$, it will at once appear that $u'$ will not become infinite so long as $y''$ is finite—that is, so long as there occur no cusps. Were this not so, we could not feel entire confidence in the following investigations.

But in order to make $u$ vanish without supposing either of the quantities $\frac{dy}{dc_1}$ or $\frac{dy}{dc_2}$ to vanish, we must have

$$x - \frac{y}{y'} = c_2 - L. \tag{22}$$

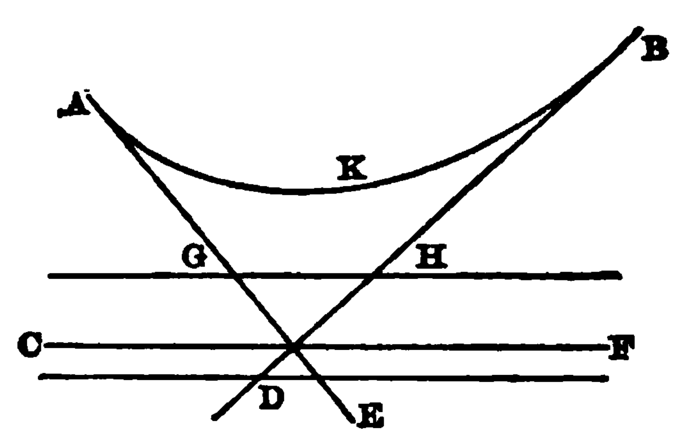
Now if $y$ be the ordinate of the curve, we know that the first member of (22) will represent the abscissa of the point in which the tangent to the curve at $y$ will meet the axis of $x$, and we will denote this abscissa by $X$. But since $L$ is a constant entirely in our power, we can give to the second member of (22) any value we please. If, therefore, there be any real value which $X$ cannot assume, we can, by making the second member take that value, render equation (22) impossible, and thus cause that $u$ shall not vanish at all.

But suppose either of the quantities $\frac{dy}{dc_1}$ or $\frac{dy}{dc_2}$ to vanish twice. Then equating the first to zero, we obtain $x - \frac{y}{y'} = c_2$. Whence, if it vanish twice, there must be two tangents which meet on the axis of $x$ at the point whose abscissa is $c_2$. That the second quantity may vanish twice, $y'$ must also vanish twice.

**147.** We may now complete the discussion of Prob. VIII., as promised in the closing remark of Art. 63.

Here $n$ is unity, and $f$ of that article is identical with $v$. Suppose, as before, that $y$ is positive, but that the curve, instead of being concave, is always convex to the axis of $x$. Then $X$ cannot always range over all real values. For sup-

pose the line $AE$ to slide as a tangent along the curve from $A$ to $B$. Then if we assume $DE$ as the axis of $x$, this line cannot meet $x$ between $D$ and $E$, and the range of $X$ is not therefore complete. But if $CF$ be the axis of $x$, $X$ will assume all real values, its range being just complete; while if $GH$ be taken as the axis of $x$, then $X$, having passed through infinity, will complete its range before $B$ is reached, and will then repeat the values of $x$ from $G$ to $H$. If we consider such an arc as $BK$, the range of $X$ will evidently be restricted, and the tangents at $B$ and $K$ will intersect above $K$—that is, above $x$—since the ordinate of $K$ must be positive.




Hence when $y''$ is positive, if the tangents at the extremities of the arc intersect above the axis of $x$, we shall have a maximum or a minimum according as $v''$ is negative or positive, because $y$ is positive, and we have seen (Art. 63) that when $y''$ is of invariable sign, $f''$, which is here $v''$, will be also. But if the extreme tangents intersect on or below the axis of $x$, there can be neither a maximum nor a minimum.

## Problem XXVIII.

**148.** *It is required by means of the preceding method to apply Jacobi's Theorem to Prob. VII.*

Here the general equation to be considered is

$$U = \int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx = \int_{x_0}^{x_1} yv\,dx.$$

Whence $v'' = \frac{1}{\sqrt{(1+y'^2)^3}}$, a positive quantity; and as the general solution is a catenary, having the directrix as the axis of $x$, $y''$ is always positive. Therefore we infer that the solution will render $U$ a minimum when the extreme tangents intersect above the axis of $x$, but not otherwise.

Suppose, then, the same condittons and notation as in Art. 61, which will of course hold even should $y_0$ and $y_1$ become equal. Now the equations of the extreme tangents are

$$y - b = y_1'(x-c) \qquad \text{and} \quad y - k = y_0'(x+c).$$

From these equations we obtain

$$\frac{y_1'}{y_0'} = \frac{y - b + y_1'c}{y - k - y_0'c};$$

and solving for $y$, and giving it a suffix, because it will then be the ordinate of the point of intersection only, we have

$$y_2 = \frac{2cy_0'y_1' + ky_1' - by_0'}{y_1' - y_0'}. \tag{1}$$

Now put

$$L = e^{\frac{2c}{a}} - e^{-\frac{2c}{a}}. \tag{2}$$

Then multiply equation (4), Art. 61, by $\frac{1}{L}e^{\frac{x}{a}}$, equation (5) by $\frac{1}{L}e^{-\frac{x}{a}}$, subtracting the second product from the first, and then, observing that the first member of the resulting equation becomes identical with the second member of equation (1) of the same article, we have, as the equation of the catenary,

$$y = \frac{1}{L}\left\{e^{\frac{x}{a}}\left(be^{\frac{c}{a}} - ke^{-\frac{c}{a}}\right) + e^{-\frac{x}{a}}\left(ke^{\frac{c}{a}} - be^{-\frac{c}{a}}\right)\right\}. \tag{3}$$

Now differentiating (3) with respect to $x$ only, and then substituting successively in the result $e^{\frac{c}{a}}$ and $e^{-\frac{c}{a}}$ for $e^{\frac{x}{a}}$, we have

$$y_1' = \frac{Mb - 2k}{La}, \qquad (4)$$

$$y_0' = \frac{2b - Mk}{La}, \qquad (5)$$

where

$$M = e^{\frac{2c}{a}} + e^{-\frac{2c}{a}}. \qquad (6)$$

Therefore

$$y_1' - y_0' = (M - 2)(b + k)\frac{1}{La}. \qquad (7)$$

But

$$L^2 = M^2 - 4 = (M + 2)(M - 2). \qquad (8)$$

Whence $M - 2$ must be positive; and as $L$ cannot become negative, (7) must also be positive. Multiplying (4) by $k$, (5) by $b$, and subtracting, and then multiplying (4) by (5), we have the equations

$$ky_1' - by_0' = \frac{2Mbk - 2(b^2 + k^2)}{La} \qquad (9)$$

and

$$y_0'y_1' = \frac{2M(b^2 + k^2) - bk(4 + M^2)}{L^2a^2}$$

$$= \frac{2M(b^2 + k^2) - bk(4 + 2M^2 - M^2)}{L^2a^2}. \qquad (10)$$

Multiplying (10) by $2c$, adding to (9), reducing to a common denominator, and factoring, we have

$$2c\,y_0'y_1' + ky_1' - by_0' =$$

$$\frac{2}{L^2a^2}\left\{(b^2 + k^2 - Mbk)(2Mc - La) + (M^2 - 4)\,cbk\right\}. \qquad (11)$$

But performing the multiplication indicated in the second member of equation (6), Art. 61, it may be written

$$\frac{L^2a^2}{4} = Mbk - (b^2 + k^2). \tag{12}$$

Hence, and recollecting that $M^2 - 4 = L^2$, the second member of (11) will become

$$\frac{La}{2} - Mc + \frac{2cbk}{a^2}. \tag{13}$$

But equation (8), Art. 61, may be written

$$F' = L\left\{\frac{La}{2} - Mc + \frac{2cbk}{a^2}\right\}; \tag{14}$$

and hence, since $L$ is always positive, the sign of (13), and consequently that of $y_2$, the ordinate of the point in which the extreme tangents intersect, will be like that of $F'$.

Now it was shown that when but one catenary can be drawn, $F'$ is zero; but that when two catenaries can be drawn, $F'$ will be positive for the upper and negative for the lower. Hence the extreme tangents to the upper catenary will intersect above the axis of $x$, thus giving us a minimum; while those to the lower will intersect below that axis, and will not give a minimum. When but one catenary can be drawn, the extreme tangents will intersect on the directrix, and we shall not have a minimum. Indeed, we may here suppose that the two catenaries coincide; and for a demonstration of the fact that the extreme tangents would in this case intersect on the directrix, see Todhunter's Researches, Art. 72.

## Problem XXIX.

**149.** *It is required to apply the general method of Art.* 146 *to Case* 2, *Prob. II.*

Here $n = -\frac{1}{2}$ and $v = \sqrt{1 + y'^2}$, so that

$$\frac{d^2V}{dy'^2} = y^n v'' = \frac{1}{\sqrt{y(1 + y'^2)^3}},$$

which is always positive and finite; thus indicating a minimum, so far as it is concerned. Now as the general solution in this case is a cycloid, having the horizontal as the axis of $x$, we know that $X$ cannot assume all possible values, since no tangent can meet the axis of $x$ within the cycloid. Hence, without determining $y$ as a function of $x$, or even obtaining the value of $u$, we are able easily to apply the method of Jacobi, and to see that we have a minimum.

This result is, however, subject to any doubt which may arise from the fact that $y'$ is infinite at either cusp, but is altogether trustworthy so long as the portion of the curve which we are considering does not contain any cusp, as will be the case if the particle is to start with an initial velocity.

## Problem XXX.

**150.** *It is required to apply the theorem of Jacobi to Prob. XVI.*

Here, as will be seen from equation (8), Art. 98, the general solution is a sphere, having its centre upon the axis of $x$; and, recollecting that $y$ must not become negative, that equation may be written

$$y = \sqrt{4a^2 - (x - c_2)^2}. \tag{1}$$

Now it must be observed that $a$ is not a constant of integration, but was introduced in accordance with Euler's method for treating problems of relative maxima and minima, so that it cannot be varied in applying Jacobi's Theorem; and functions involving it, together with $x$, $y$, $c_1$ and $c_2$, will merely be mentioned as functions of the latter quantities.

It appears, then, that $y$ has in this case been obtained merely as a function of $x$ and $c_2$, it having been necessary in equation (3), Art. 98, to make the first constant of integration zero before we could effect the second integration. Since, therefore, the constant $c_1$ has disappeared from the value of $y$, we shall not be able readily to obtain the functions $\frac{dy}{dc_1}$ and $\frac{dy}{dc_2}$ required in the application of Jacobi's Theorem.

**151.** Since we have seen (Art. 99) that the sign of $2a$ must be negative, we have from equation (1), Art. 98,

$$V = y^2 - 2ay\sqrt{1+y'^2}.$$

Therefore

$$\frac{d^2V}{dy'^2} = -\frac{2ay}{\sqrt{(1+y'^2)^3}},$$

which, being negative, indicates, so far as it is concerned, that the volume is a maximum.

Now observing the sign of $2a$, equation (3), Art. 98, may be written

$$\frac{2ay}{\sqrt{1+y'^2}} = y^2 + c_1. \tag{2}$$

But from (2) we see that $y'$ can be expressed as an explicit function of $y$ and $c_1$; and we have always

$$dx \text{ or } \frac{dy}{y'} = f'(y, c_1)\,dy. \tag{3}$$

Whence, supposing the integration to have been performed, we have

$$x = f(y, c_1) + c_2 = f + c_2. \tag{4}$$

Therefore $\frac{df}{dy}$ must in any case equal $\frac{1}{y'}$. Taking the total differential of (4) with respect to $c_1$, recollecting that any change in $c_1$ will affect $y$ but not $x$, we have

$$0 = \frac{df}{dc_1} + \frac{df}{dy}\frac{dy}{dc_1} = \frac{df}{dc_1} + \frac{1}{y'}\frac{dy}{dc_1}. \tag{5}$$

Hence

$$\frac{dy}{dc_1} = -y'\frac{df}{dc_1}. \tag{6}$$

Now in like manner, recollecting that $c_2$ does not occur explicitly in $f$, we have

$$0 = \frac{df}{dy}\frac{dy}{dc_2} + 1 = \frac{1}{y'}\frac{dy}{dc_2} + 1,$$

and therefore

$$\frac{dy}{dc_2} = -y'. \tag{7}$$

We must now determine the value of $\frac{df}{dc_1}$, observing that it is only the partial differential coefficient of $f$ with respect to $c_1$. If $f$ could be found as an explicit function of $y$ and $c_1$, this could be done directly; but as $f$ cannot be so found, we must adopt an indirect method. Now the supposition that $y$ is to become constant, and $c_1$ variable, will make $dy$ constant, but $y'$ still variable, because it is capable of being expressed as an explicit function of $y$ and $c_1$, although $\frac{dy'}{dc_1}$ will be no longer total, but merely partial, and can be at once found. But

$f=\int\frac{dy}{y'}$; and if in this expression we vary $c_1$, regarding $y$ as constant, and indicate partial differentials by brackets, we shall have

$$\left[\frac{df}{dc_1}\right]\delta c_1 = -\int\frac{1}{y'^2}\,\delta y'\,dy.$$

But in this case we must have $\delta y'=\left[\frac{dy'}{dc_1}\right]\delta c_1$; and as $\delta c_1$ must be constant, we have

$$\left[\frac{df}{dc_1}\right] = -\int\frac{1}{y'^2}\left[\frac{dy'}{dc_1}\right]dy. \tag{8}$$

Now from (2), by partial differentiation, we obtain

$$-\frac{2ayy'}{\sqrt{(1+y'^2)^3}}\left[\frac{dy'}{dc_1}\right] = 1. \tag{9}$$

Hence

$$\left[\frac{df}{dc_1}\right] = \frac{1}{2a}\cdot\int\frac{\sqrt{(1+y'^2)^3}}{yy'^2}\,dy. \tag{10}$$

**152.** When the general solution is a sphere, this integral can be obtained. For if in (2) we put $r$ for $2a$, make $c_1$ zero, and divide by $y$, it will become the differential equation of the circle, whose centre is on the axis of $x$; and we shall have

$$y=\frac{r}{\sqrt{1+y'^2}} \qquad \text{and} \qquad dy=-\frac{ry'dy'}{\sqrt{(1+y'^2)^3}}.$$

Hence (10) becomes

$$\left[\frac{df}{dc_1}\right] = -\frac{1}{r}\int\frac{\sqrt{1+y'^2}}{y'^2}\,dy'. \tag{11}$$

Now put $y' = \tan w$ and $dy' = \frac{dw}{\cos^2 w}$. Then

$$\left[\frac{df}{dc_1}\right] = -\frac{1}{r}\int \frac{dw}{\cos w \sin^2 w} = -\frac{1}{r}\int \frac{\cos^2 w + \sin^2 w}{\cos w \sin^2 w} dw$$

$$= -\frac{1}{r}\left\{\int \frac{\cos w}{\sin^2 w} dw + \int \frac{dw}{\cos w}\right\}. \tag{12}$$

Now by integrating this expression, we shall obtain

$$\left[\frac{df}{dc_1}\right] = -\frac{1}{r}\left\{-\frac{1}{\sin w} + \frac{1}{2} l \frac{1 + \sin w}{1 - \sin w}\right\} = -\frac{1}{r} Z. \tag{13}$$

Hence, finally, by equation (6), we have

$$\frac{dy}{dc_1} = \frac{y'}{r} Z. \tag{14}$$

It will then at once appear, by comparing (7) and (14), that the range which we are in this case to examine will be entirely dependent upon that of $Z$. Now when $w$ is $\frac{\pi}{2}$, $Z$ is $-\infty$; and when $w$ is zero, $z$ is $+\infty$; so that $Z$ ranges twice from $-\infty$ to $+\infty$ as we pass from $x_0$ to $x_1$. We would therefore naturally infer, from the employment of Jacobi's method, that the sphere is not the solid of revolution whose volume for a given surface is a maximum; an inference which we know to be erroneous.

**153.** Although for convenience we have hitherto tacitly assumed that, even when the terms of the second order are to be considered, we may by Euler's method convert any problem of relative maxima or minima into one of absolute maxima or minima, we have not yet established the correctness of this assumption; while we see from the last article that it can-

not be universally true. In order to discuss the subject in a general manner, let us resume the conditions and notation at the beginning of Art. 92. Then, as there, we shall have

$$\int_{x_0}^{x_1} \delta v \, dx = \int_{x_0}^{x_1} V \delta y \, dx \quad \text{and} \quad \int_{x_0}^{x_1} \delta v' \, dx = \int_{x_0}^{x_1} V' \delta y \, dx.$$

Moreover, since the limiting values of $\delta y$, $\delta y'$, etc., are to vanish, the terms of the second order will become

$$\frac{1}{2}\int_{x_0}^{x_1} \delta V \delta y \, dx \qquad \text{and} \quad \frac{1}{2}\int_{x_0}^{x_1} \delta V' \delta y \, dx.$$

This we have already seen to be the case when the function contains no differential coefficient higher than $y'$, and we shall subsequently see that it is true generally.

It must likewise be observed that now, besides being infinitesimal, the variations of $y$, $y'$, etc., are restricted to such systems of values as will render $\int_{x_0}^{x_1} v' dx$ constant; and although we cannot express explicitly the nature of this restriction, and although the systems of values which it permits for $\delta y$, $\delta y'$, etc., may still be infinite in number, it cannot be disregarded in the discussion of the problem.

We shall denote this restriction by writing the variations affected in brackets; then, to the second order, we have

$$\int_{x_0}^{x_1} [\delta v] \, dx = \int_{x_0}^{x_1} V[\delta y] \, dx + \frac{1}{2}\int_{x_0}^{x_1} [\delta V] \, [\delta y] \, dx = k + l \qquad (1)$$

and

$$\int_{x_0}^{x_1} [\delta v'] \, dx = \int_{x_0}^{x_1} V'[\delta y] \, dx + \frac{1}{2}\int_{x_0}^{x_1} [\delta V'] \, [\delta y] \, dx = m + n. \quad (2)$$

Now since $\int_{x_0}^{x_1} v \, dx$ is to be a relative maximum or minimum, $k + l$ must certainly be a small negative or positive quantity

of the second order; and since $\int_{x_0}^{x_1} v' dx$ is to undergo no change when $y, y'$, etc., are varied, $m + n$ must vanish, at least so far as any quantity of the second order is concerned.

**154.** Thus far there can be no doubt; but what follows may perhaps be subject to some criticism, as the author has not seen it in any other work, although he will not assert that no similar discussion occurs.

Now the equation $m = -n$ must be true to the second order, so that it appears that $m$ need not vanish absolutely, but must become less than any quantity of the first order; and we are therefore led to infer that $k$ also will not vanish, but become a quantity of the second order. That this supposition is not inadmissible in problems of relative maxima and minima, we have already seen in the beginning of Art. 94. But these suppositions regarding $k$ and $m$ will not invalidate the reasoning of Art. 92, by which it was shown that $f$ or $\frac{V'}{V}$ must be a constant; because $f$ could not differ from a constant by any finite quantity.

Now assume the equation

$$\int_{x_0}^{x_1} v\,dx + b\int_{x_0}^{x_1} v'dx = u, \tag{3}$$

where $b$ is any constant whatever. Then, since $\int_{x_0}^{x_1} v'\,dx$ is to undergo no change when we vary $y, y'$, etc., the variation of $u$ to any order, as the second, will to that order equal merely the variation of its first term. Hence we may write

$$\int_{x_0}^{x_1} [\delta v]\,dx = \int_{x_0}^{x_1} \left\{ V + bV' \right\} [\delta y]\,dx$$

$$+ \frac{1}{2}\int_{x_0}^{x_1} \left\{ [\delta V] + b[\delta V'] \right\} [\delta y]\,dx = k + bm + l + bn. \tag{4}$$

Now so long as $b$ remains undetermined, $k+bm$ may be a quantity of the second order; but when, as explained in Art. 92, we put $b=a=-\frac{V}{V'}$, we effect that $k+am$ shall certainly vanish, since those terms are then equivalent to

$$\int_{x_0}^{x_1}\left\{V-V\right\}[\delta y]\,dx.$$

Therefore we have

$$l+an=\frac{1}{2}\int_{x_0}^{x_1}\left\{[\delta V]+a[\delta V']\right\}[\delta y]\,dx, \qquad (5)$$

as the exact expression to the second order of the change which $\int_{x_0}^{x_1}v\,dx$ will experience when $y, y'$, etc., are varied according to the conditions of the problem; and this is the only mode of rendering the expression exact, since it is not only sufficient, but also necessary, that $b$ should become $a$ in order to make the terms of the first order entirely vanish.

Now according to Euler's method, let $U$ be what $u$ becomes when $b=a$. Then to the second order we have

$$\left[\delta\int_{x_0}^{x_1}v\,dx\right]=[\delta U]=\frac{1}{2}\int_{x_0}^{x_1}\left\{[\delta V]+a[\delta V']\right\}[\delta y]\,dx. \qquad (6)$$

Whence it appears that we can and must employ Euler's method to obtain the terms of the second order in an explicit form. But it will be observed that the restriction still adheres to the variations in (6), and no method of further determining its effect upon the general form of $\delta U$ has yet been devised; still, if, as is usually the case, the general solution can render the second member of (6) invariably negative or positive for unrestricted values of $\delta y$, $\delta y'$, etc., this restriction can, of course, exercise no influence upon the problem, and we shall be certain of a maximum or a minimum. But if, on the other

hand, by employing the most general values of $\delta y$, $\delta y'$, etc., it should be found possible to cause the second member of (6) to assume either sign or to vanish, we may conclude justly that $U$ is not an absolute maximum or minimum. But this will not warrant us in asserting that $U$, and consequently $\int_{x_0}^{x} v\,dx$, may not be a relative maximum or minimum; that is, a maximum or minimum for all such values of $\delta y$, $\delta y'$, etc., as will render $\int_{x_0}^{x_1} v'dx$ constant; and having no means of taking proper account of this restriction upon the variations, we may, at least theoretically, be unable to determine whether $U$ is or is not a relative maximum or minimum.

**155.** Thus we see, first, that Euler's method must be employed in developing the terms of the second order in this class of problems; and if by it we seem to have a maximum or a minimum, we may accept the decision as final. But if, on the contrary, we appear to have neither a maximum nor a minimum, we cannot always conclude that such is really the case, the discrimination being correct as regards an absolute, but perhaps not as regards a relative maximum or minimum state of $U$.

This latter result is mentioned by Prof. Todhunter (see his Researches, Art. 283); and evident as it is, when the former is admitted, it appears not to have been noticed by any previous writer. The former result, however, is assumed by him without proof. Prof. Jellett has given no discussion of the terms of the second order in questions of this character.

**156.** We can now understand why the theorem of Jacobi is not as satisfactory for problems of relative as for those of absolute maxima and minima. For example, in the preceding problem the condition that the surface is to remain constant will prevent us from making $\delta y$ invariably positive or negative; and as it must change sign, it will certainly vanish at

least once between the limits $x_0$ and $x_1$, say at the point whose co-ordinates are $x_2$ and $y_2$. But even if we can so select $x_2$ that $u$ can vanish both when $x = x_0$ and $x = x_2$, as we certainly can by considering a hemisphere, it does not follow that we can make the terms of the second order throughout the integral vanish by the use of $ku$. For when we assume $\delta y = ku$ throughout the first hemisphere, we may be obliged to make some change in the form of the other also; that is, $ku$ may not be an admissible value of $\delta y$ unless the first hemisphere be permitted to increase or diminish its surface.

Nevertheless, when Jacobi's method seems to indicate a maximum or a minimum, that indication may be regarded as trustworthy.

**157.** We may, in passing, notice two particular and exceptional cases which may arise in the general application of this theorem. These cases appear to have been first noticed by Spitzer. (See Todhunter's History of Variations, Arts. 173, 174.) Suppose, first, that $\frac{d^2V}{dy'^2} = 0$ throughout the integral. Now if $V$ involve $y'$ at all, it can, to render this equation true, contain only its first power. Therefore the general form of $V$ must be

$$V = f(x, y) + y'F(x, y) = f + y'F. \tag{1}$$

We shall write total differentials in brackets. Then

$$U = \int_{x_0}^{x_1} V dx,$$

the limiting values of $x$ and $y$ being fixed; and therefore to the first order we have

$$\delta U = \int_{x_0}^{x_1} \left\{ \left(\frac{df}{dy} + y'\frac{dF}{dy}\right) \delta y + F\delta y' \right\} dx = 0. \tag{2}$$

Therefore, as usual, we obtain

$$\frac{df}{dy}+y'\frac{dF}{dy}-\left[\frac{dF}{dx}\right]=0. \qquad (3)$$

But

$$\left[\frac{dF}{dx}\right]=\frac{dF}{dx}+y'\frac{dF}{dy},$$

so that (3) becomes

$$\frac{df}{dy}-\frac{dF}{dx}=0. \qquad (4)$$

Now (4) involving only $x, y$, and possibly constants, which are not of integration, we can, by solving for $y$, obtain it as a function of $x$ without constants of integration. Hence, in applications to geometry, it will be impossible to satisfy the general solution unless the given points happen to be situated upon the curve which is determined by (4).

The second case is that in which we have

$$\frac{d^2V}{dy'^2}=0, \qquad \text{and} \qquad \frac{d^2V}{dy^2}-\left[\frac{d}{dx}\frac{d^2V}{dydy'}\right]=0.$$

As this case is more difficult than the former, and is rather curious than important, we shall merely give its interpretation without proof.

First, $f$ being some function of $x$ and $y$, $f'$ and $f''$ being functions of $x$ only, and the differentials not enclosed in brackets being partial, it is shown that $V$ must have the general form

$$V=\frac{df}{dx}+y'\frac{df}{dy}+yf'+f''.$$

Whence

$$U=\int_{x_0}^{x_1}Vdx=f_1-f_0+\int_{x_0}^{x_1}yf'dx+\int_{x_0}^{x_1}f''dx.$$

Therefore

$$\delta U = \left(\frac{df}{dy}\right)_1 \delta y_1 - \left(\frac{df}{dy}\right)_0 \delta y_0 + \int_{x_0}^{x_1} f' \delta y \, dx = 0.$$

Hence if $U$ is to be a maximum or a minimum, $f'$ must vanish for all values of $x$, and $U$ must be of the general form

$$U = f_1 - f_0 + \int_{x_0}^{x_1} f'' \, dx,$$

which, since the last integral is constant and might be written $F(x)$, is not a general problem of variations. Thus in this case the maximum or minimum value of $U$ must be sought, if at all, by the differential calculus; and if the limiting values of $x$ and $y$ be fixed, $U$ can have no maximum or minimum state.

In both these cases $V$ involves the first power only of $y'$, and they are therefore examples of Exception 2, Art. 51.

**158.** We may now, before considering the next case, present the following general view of the treatment of the terms of the second order according to Jacobi.

Assume the equation $U = \int_{x_0}^{x_1} V dx$, where $V$ is any function of $x$, $y$, $y'$ . . . . $y^{(n)}$, and regard the limiting values of $x, y, y'$ . . . . $y^{(n-1)}$ as fixed. Then, as before, the solution must be obtained from the differential equation $M = 0$, which will in general be of the order $2n$. Hence its complete integral will involve $2n$ arbitrary constants, and may be written

$$y = f(x, c_1, c_2, c_3 \, . \, . \, . \, . \, c_{2n}) = f,$$

and this solution is rendered complete when the constants are so determined as to satisfy the conditions at the limits.

**159.** Next the terms of the second order must equal

$$\frac{1}{2} \int_{x_0}^{x_1} \delta M \, \delta y \, dx.$$

For we have always

$$\delta V = \frac{dV}{dy}\delta y + \frac{dV}{dy'}\delta y' + \text{etc.} \ldots + \frac{dV}{dy^{(n)}}\delta y^{(n)}.$$

But if we vary these coefficients, leaving $\delta y$, $\delta y'$, etc., unvaried, we shall obtain the well-known form for the terms of the second order in $\delta V$; namely,

$$\frac{d^2V}{dy^2}\delta y^2 + \frac{2d^2V}{dy\,dy'}\delta y\,\delta y' + \frac{d^2V}{dy'^2}\delta y'^2 + \text{etc.} \ldots + \frac{d^2V}{dy^{(n)2}}\delta y^{(n)2}.$$

Therefore it appears that the terms of the second order in $\delta U$ must in any case equal half of what would result from varying those of the first order, supposing $\delta y$, $\delta y'$, etc., to undergo no change. They should not, however, be considered as really arising in this manner, as $y$, $y'$, etc., receive no second increment. But when the limiting values of $y$, $y'$, etc., are fixed, the terms of the first order in $\delta U$ become $\int_{x_0}^{x_1} M\,\delta y\,dx$, so that those of the second order must equal $\frac{1}{2}\int_{x_0}^{x_1}\delta M\,\delta y\,dx$.

**160.** It is evident that the reasoning of Art. 132 would be equally applicable whatever might be the order of the differential equation $M = 0$, and we shall therefore assume at once that $\delta M$ and $\int_{x_0}^{x_1}\delta M\,\delta y\,dx$ will vanish if for $\delta y$ we substitute

$$u = \frac{df}{dc_1}\delta c_1 + \frac{df}{dc_2}\delta c_2 + \text{etc.} \ldots + \frac{df}{dc_{2n}}\delta c_{2n},$$

the variations of $c_1$, $c_2$, etc., being, as before, entirely independent. Then $\delta y'$, $\delta y''$, etc., will become $\frac{du}{dx}$ or $u'$, $\frac{d^2u}{dx^2}$ or $u''$, etc., the differentials being total with respect to $x$.

It will next be shown that $\delta M$ can be made to assume the form

$$\delta M = A\delta y + \frac{d}{dx} A_1 \delta y' + \text{etc.} \ldots + \frac{d^n}{dx^n} A_n \delta y^{(n)}.$$

After this the terms of the second order can be integrated by parts, until they finally take the form $\delta U = \frac{1}{2}\int_{x_0}^{x_1} \frac{d^2 V}{dy^{(n)2}}$ multiplied by the square of a certain function, analogous to that previously found.

As the proof of the last two points is necessarily difficult, the general reader may, without serious loss, omit the remainder of this theorem, or may at least assume the truth of the two following lemmas, whose use will be at once evident.

*Lemma I.*

**161.** $\delta M$ can always be put under the form

$$\delta M = A\delta y + \frac{d}{dx} A_1 \delta y' + \text{etc.} \ldots, + \frac{d^n}{dx^n} A_n \delta y^{(n)}.$$

We shall, for convenience, abandon our former notation, and, adopting that of Prof. Jellett, write

$$M = N - \frac{dP_1}{dx} + \frac{d^2 P_2}{dx^2} - \text{etc.} \ldots \pm \frac{d^n P_n}{dx^n}. \qquad (1)$$

Whence

$$\delta M = \delta N - \delta \frac{dP_1}{dx} + \text{etc.} \ldots \pm \delta \frac{d^n P_n}{dx^n}$$

$$= \delta N - \frac{d\delta P_1}{dx} + \text{etc.} \ldots \pm \frac{d^n \delta P_n}{dx^n}. \qquad (2)$$

For take any term as $\delta\dfrac{d^m P_m}{dx^m} = \delta t^{(m)}$, where $t = P_m$. Now if in Art. 9, we put $t$ for $y$, $t'$ for $y'$, etc., recollecting that $t'$, $t''$, etc., are the total differential coefficients of $t$ with respect to $x$, we shall, by reasoning precisely like that there employed, find that $\delta t^{(m)} = \dfrac{d^m \delta t}{dx^m}$; so that it is evident that (2) has been correctly transformed. But

$$\delta P_m = \frac{dP_m}{dy}\,\delta y + \frac{dP_m}{dy'}\,\delta y' + \text{etc.} \ldots + \frac{dP_m}{dy^{(n)}}\,\delta y^{(n)},$$

and

$$P_m = \frac{dV}{dy^{(m)}}.$$

Therefore

$$\delta P_m = \frac{d^2 V}{dy^{(m)}dy}\,\delta y + \frac{d^2 V}{dy^{(m)}dy'}\,\delta y' + \text{etc.} \ldots + \frac{d^2 V}{dy^{(m)}dy^{(n)}}\,\delta y^{(n)}.$$

Hence

$$\frac{d^m\,\delta P_m}{dx^m} = \frac{d^m}{dx^m}\left\{\frac{d^2 V}{dy^{(m)}dy}\,\delta y + \frac{d^2 V}{dy^{(m)}dy'}\,\delta y' + \text{etc.} \ldots + \frac{d^2 V}{dy^{(m)}dy^{(n)}}\,\delta y^{(n)}\right\}.$$

Now consider some individual term of this series, as

$$\frac{d^m}{dx^m}\,\frac{d^2 V}{dy^{(m)}\,dy^{(l)}}\,\delta y^{(l)} = \frac{d^m}{dx^m}\,k\,\delta y^{(l)}, \tag{3}$$

where $l$ is not greater than $m$, and $k = \dfrac{d^2 V}{dy^{(m)}\,dy^{(l)}}$. Now if $l$ equal $m$, this term is already under the required form; but if $l$ be less than $m$, there will certainly arise from the development of $\dfrac{d^l P_l}{dx^l}$ a term of the form $\dfrac{d^l}{dx^l}\,k\,\delta y^{(m)}$, the sign of these

terms being like or unlike, according as $m - l$ is even or odd. That is, if $\delta M$ be fully written out, it will be found that with the exception of those terms which are already under the required form, all the others may be arranged in pairs, the type of which is the pair

$$\frac{d^m}{dx^m} k\,\delta y^{(l)} \pm \frac{d^l}{dx^l} k\,\delta y^{(m)}. \tag{4}$$

But by a theorem of the differential calculus, any pair of the form (4) can be arranged in a series of the form

$$\frac{d^l}{dx^l} c_l\,\delta y^{(l)} + \frac{d^{l+1}}{dx^{l+1}} c_{l+1}\,\delta y^{(l+1)} + \text{etc.}$$

(See Note to Lemma I.)

Whence it appears that all the terms in $\delta M$ can be arranged as stated at the beginning of this lemma.

*Lemma II.*

**162.** If $A$, $A_1$, etc., be functions of $x$, implicit or explicit, and $u$ any quantity which will satisfy the equation

$$Au + \frac{d}{dx} A_1 u' + \frac{d^2}{dx^2} A_2 u'' + \text{etc.} = 0; \tag{1}$$

then if we write

$$U = u \left\{ Aut + \frac{d}{dx} A_1 (ut)' + \frac{d^2}{dx^2} A_2 (ut)'' + \text{etc.} \right\}, \tag{2}$$

$Udx$ will always be integrable whatever be the value of $t$, the integral taking the form

$$\int U dx = B_1 t' + \frac{d}{dx} B_2 t'' + \text{etc.}, \tag{3}$$

where $B_1$, $B_2$, etc., are functions derived from $A$, $A_1$, etc.

As the proof of this theorem belongs entirely to the integral calculus, we follow the plan of Prof. Jellett, and append it in a note (see Note to Lemma II.).

## CASE 2.

**163.** Next let $U = \int_{x_0}^{x_1} V dx$, where $V$ is any function of $x, y, y'$ and $y''$, the limiting values of $x$, $y$ and $y'$ being fixed.

Then, proceeding in the usual manner, the general solution must be found from the differential equation $M = 0$, where

$$M = \frac{dV}{dy} - \frac{d}{dx}\frac{dV}{dy'} + \frac{d^2}{dx^2}\frac{dV}{dy''}. \tag{1}$$

The complete integral of (1) will give $y$ in the form

$$y = f(x, c_1, c_2, c_3, c_4) = f, \tag{2}$$

in which the four constants must be so determined as to satisfy the given values of $y_1, y_0, y_1', y_0'$.

But when these limiting values are fixed, we need not express the terms of the second order in the usual way, which expression would be difficult to transform; but we may write at once

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \delta M\, \delta y\, dx. \tag{3}$$

We have now an invariable method of transforming $\delta U$, since we can always, according to Lemma I., put $\delta M$ under the form

$$\delta M = A\delta y + \frac{d}{dx}A_1\delta y' + \frac{d^2}{dx^2}A_2\delta y'',$$

and we shall now proceed to apply this lemma in order to determine the functions $A$, $A_1$ and $A_2$.

**164.** For brevity of notation, let $a_{yy}$, $a_{yy'}$, $a_{y'y'}$, $a_{yy''}$, $a_{y'y''}$ and $a_{y''y''}$ denote the second differentials of $V$ with regard respectively to $y, y$ and $y', y', y$ and $y'', y'$ and $y''$, and $y''$. Then, referring to the value of $M$ in (1), and writing its variation in full, recollecting that the variation of the differential of any quantity equals the differential of the variation of that quantity, we have

$$\delta M = a_{yy}\delta y + a_{yy'}\delta y' + a_{yy''}\delta y'' - \frac{d}{dx}(a_{y'y}\delta y + a_{y'y'}\delta y' + a_{y'y''}\delta y'')$$

$$+ \frac{d^2}{dx^2}(a_{y''y}\delta y + a_{y''y'}\delta y' + a_{y''y''}\delta y'')$$

$$= a_{yy}\delta y - \frac{d}{dx}a_{y'y'}\delta y' + \frac{d^2}{dx^2}a_{y''y''}\delta y'' + \left(-\frac{d}{dx}a_{yy'}\delta y + a_{yy'}\delta y'\right)$$

$$+ \left(\frac{d^2}{dx^2}a_{yy''}\delta y + a_{yy''}\delta y''\right) + \left(\frac{d^2}{dx^2}a_{y'y''}\delta y' - \frac{d}{dx}a_{y'y''}\delta y''\right)$$

$$= a_{yy}\delta y + \frac{d}{dx}\left(-a_{y'y'}\delta y'\right) + \frac{d^2}{dx^2}a_{y''y''}\delta y'' + \left(\frac{d}{dx}k_1\delta y - k_1\delta y'\right)$$

$$+ \left(\frac{d^2}{dx^2}k_2\delta y + k_2\delta y''\right) + \left(\frac{d^2}{dx^2}k_3\delta y' - \frac{d}{dx}k_3\delta y''\right), \tag{4}$$

where

$$k_1 = -a_{yy'}, \qquad k_2 = a_{yy''}, \qquad k_3 = a_{y'y''}. \tag{5}$$

Now the first three terms of (4) are already in the required form, so that, setting these aside, we will consider the first couple. Here $l = 0$, $n = 1$, and there can be but one term resulting from this pair. Therefore, by equation (13), Note to Lemma I., the couple becomes

$$\frac{d^l}{dx^l}c_l\,\delta y^{(l)}, \quad \text{or} \quad c_l\,\delta y, \quad \text{or} \quad \frac{d^n}{dx^n}ak_1.\delta y,$$

$$\text{or} \quad \frac{d}{dx}ak_1.\delta y, \quad \text{or} \quad \frac{d}{dx}(-a_{yy'}).\delta y, \tag{6}$$

because $a$ is always unity. Now consider the next couple. Here $l = 0$, $n = 2$, and the number of terms which will result is two. Hence, by (13), the pair becomes

$$c_0\,\delta y + \frac{d}{dx} c_1 \delta y'.$$

We also have by equation (14) of the same note, since $a$ is always one, and $b$ is in this case two,

$$c_0 = \frac{d^2 k_2}{dx^2}, \qquad c_1 = 2k_2,$$

and

$$c_0 \delta y + \frac{d}{dx} c_1 \delta y' = \frac{d^2 k_2}{dx^2} \delta y + \frac{d}{dx} 2k_2 \delta y'$$

$$= \frac{d^2 a_{yy''}}{dx^2} \delta y + \frac{d}{dx} 2a_{yy''} \delta y'. \tag{7}$$

In the last pair we have $l = 1$, $n = 1$, and it becomes

$$\frac{d}{dx}\left(\frac{dk_2}{dx} \delta y'\right) = \frac{d}{dx}\left(\frac{da_{y'y''}}{dx} \delta y'\right). \tag{8}$$

Collecting results from the last members of (4), (6), (7) and (8), and arrånging, we have

$$\delta M = A \delta y + \frac{d}{dx} A_1 \delta y' + \frac{d^2}{dx^2} A_2 \delta y'',$$

where

$$A = a_{yy} - \frac{da_{yy'}}{dx} + \frac{d^2 a_{yy''}}{dx^2}, \qquad \text{or } a_{yy} - a'_{yy'} + a''_{yy''},$$

$$A_1 = -a_{y'y'} + \frac{da_{y'y''}}{dx} + 2a_{yy''}, \quad \text{or} \quad -a_{y'y'} + a'_{y'y''} + 2a_{yy''},$$

$$A_2 = a_{y''y''}. \tag{9}$$

**165.** We may now write

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \left\{ A\delta y + \frac{d}{dx} A_1 \delta y' + \frac{d^2}{dx^2} A_2 \delta y'' \right\} \delta y\, dx; \qquad (10)$$

and we know that if $u$ be an admissible value of $\delta y$, $u$ having the form given in Art. 160, $\delta U$ can be rendered zero, and we infer, as in the first case, that there will be neither a maximum nor a minimum. But since the limiting values of $y$ and $y'$ are to remain fixed, we must, in order that $\delta y$ may equal $u$, or $ku$, be able to so determine the constants $\delta c_1$, $\delta c_2$, etc., that both $u$ and $u'$ shall vanish twice simultaneously at or within the limits of integration. In the former case we may change $y$ into $y + ku$ throughout the limits, while in the latter we make this change merely for the limits at which $u$ and $u'$ vanish, leaving $y$ unvaried throughout the remainder of the integral. Also, since the variations of $y$, $y'$ and $y''$ must be infinitesimal, to make $\delta y$ equal $ku$, we must have $u$, $u'$ and $u''$ finite throughout the limits for which they are employed.

**166.** But suppose that the terms of the second order cannot be made to vanish by the use of $u$. Then if, as before, we put $ut$ for $\delta y$, (10) will become

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \left\{ Aut + \frac{d}{dx} A_1 (ut)' + \frac{d^2}{dx^2} A_2 (ut)'' \right\} ut\, dx$$

$$= \frac{1}{2}\int_{x_0}^{x_1} It\, dx, \qquad (11)$$

in which we know, from Lemma II., that $I dx$ is immediately integrable, giving

$$\int I\, dx = B_1 t' + \frac{d}{dx} B_2 t''. \qquad (12)$$

**167.** Let us next determine the functions $B_1$ and $B_2$. From (10) and (11) we have

$$Au + \frac{d}{dx}A_1 u' + \frac{d^2}{dx^2}A_2 u'' = 0 \tag{13}$$

and

$$I = \left\{ Aut + \frac{d}{dx}A_1(ut)' + \frac{d^2}{dx_2}A_2(ut)'' \right\} u. \tag{14}$$

Whence, multiplying (13) by $ut$, and subtracting from (14), we have

$$I = u\frac{d}{dx}A_1(ut)' + u\frac{d^2}{dx^2}A_2(ut)'' - ut\frac{d}{dx}A_1u' - ut\frac{d^2}{dx^2}A_2u''$$

$$= u\,\{A_1(ut)'\}' + u\,\{A_2(ut)''\}'' - ut\,(A_1u')' - ut\,(A_2u'')''. \tag{15}$$

Now we know from Note to Lemma II. that all the terms in $I$ which contain $t$ undifferentiated must eventually cancel, so that we may neglect the last two terms in (15), and may also reject all others in $t$ as they arise. We have then

$$u\{A_1(ut)'\}' = \{uA_1(ut)'\}' - u'A_1(ut)' \qquad \text{and} \qquad (ut)' = ut' + u't.$$

Whence

$$u\{A_1(ut)'\}' = (uA_1ut')' + (uA_1u't)' - u'A_1ut' - u'A_1u't.$$

Now the second and third terms of this equation can be united into one by Note to Lemma I., because here $l = 0$ and $n = 1$. But as this term would certainly contain $t$ undifferentiated, we need not perform the operation, but may reject them together with the last, retaining only

$$(u^2A_1t')'. \tag{16}$$

Again, we have

$$u\{A_2(ut)''\}'' = \{uA_2(ut)''\}'' - 2\,\{u'A_2(ut)''\}' + u''A_2(ut)''$$

and

$$(ut)'' = ut'' + 2u't' + u''t;$$

so that

$$u\{A_2(ut)''\}'' = (uA_2ut'')'' + 2(uA_2u't')'' + (uA_2u''t)'' - 2(u'A_2ut'')'$$

$$- 4(u'A_2u't')' - 2(u'A_2u''t)' + u''A_2ut'' + 2u''A_2u't' + u''A_2u''t. \quad (17)$$

Now set aside the first and fifth terms, which are already integrable; reject the last, and also the couple 6 and 8, because they could be united into one term, $n$ being 1, and that would contain $t$ undifferentiated, because $l$ is zero. Then there will remain two couples; viz., terms 2 and 4, and 3 and 7. The first, since $l = 1$ and $n = 1$, becomes

$$\{(2uA_2u')'t'\}'. \quad (18)$$

In the last couple $l = 0$, $n = 2$, and it becomes

$$(uA_2u'')''t + 2(uA_2u''t')';$$

and rejecting the first term, we have

$$2(uA_2u''t')'. \quad (19)$$

Now collecting the terms from (16), (18), (19), and the first and fifth of (17), the result can be written thus:

$$I = \{[u^2A_1 - 4u'^2A_2 + 2uA_2u'' + 2(uA_2u')']t'\}' + (u^2A_2t'')''$$

$$= (B_1t')' + (B_2t'')'', \quad (20)$$

and this by immediate integration gives (12); and

$$B_1 = u^2A_1 - 4u'^2A_2 + 2uA_2u'' + 2(uA_2u')' \quad (21)$$

and

$$B_2 = u^2A_2. \quad (22)$$

**168.** We may now integrate (11) by parts, thus:

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} It\,dx = \frac{t_1}{2}\left\{B_1 t' + (B_2 t'')'\right\}_1 - \frac{t_0}{2}\left\{B_1 t' + (B_2 t'')'\right\}_0$$

$$-\frac{1}{2}\int_{x_0}^{x_1}\left\{B_1 t' + (B_2 t'')'\right\} t'dx. \qquad (23)$$

But

$$\delta y_1 = 0, \qquad \delta y_0 = 0, \qquad t_1 = \left(\frac{\delta y}{u}\right)_1, \qquad t_0 = \left(\frac{\delta y}{u}\right)_0.$$

If, therefore, we suppose $u$ to be so taken as not to vanish at either limit, $t_1$ and $t_0$ must vanish, and we shall have

$$\delta U = -\frac{1}{2}\int_{x_0}^{x_1}\left\{B_1 t' + (B_2 t'')'\right\} t'dx. \qquad (24)$$

But we see at once that in this case the terms of the second order require still further transformation, as they are not yet in a quadratic form; and to this we now proceed.

**169.** Let $v_a$ be such a quantity as will satisfy the differential equation

$$B_1 v'_a + (B_2 v''_a)' = 0. \qquad (25)$$

Then by putting $u_a$ for $v'_a$, we have

$$B_1 u_a + (B_2 u'_a)' = 0. \qquad (26)$$

Assuming for the present that $v_a$ and consequently $u_a$ can be determined, (24) can be still further transformed. For we see from that equation that if $u_a$ were an admissible value of $t'$

throughout the limits, $\delta U$ would to the second order reduce to zero. But whatever be the value of $t'$, we may certainly represent it by $u_a t_a$, and (24) will then become

$$\delta U = -\frac{1}{2}\int_{x_0}^{x_1}\left\{B_1 u_a t_a + [B_2(u_a t_a)']'\right\} u_a t_a \, dx$$

$$= -\frac{1}{2}\int_{x_0}^{x_1} I_1 t_a \, dx, \tag{27}$$

where $I_1 dx$ is, as we shall show, immediately integrable by the note to Lemma II., its integral taking the form

$$\int I_1 \, dx = C_2 t'_a. \tag{28}$$

**170.** To find $C_2$, multiply (26) by $u_a t_a$ and subtract from the value of $I_1$ in (27). Then we shall have

$$I_1 = u_a [B_2(u_a t_a)']' - u_a t_a (B_2 u'_a)'. \tag{29}$$

But

$$u_a\{B_2(u_a t_a)'\}' = \{u_a B_2(u_a t_a)'\}' - u'_a B_2(u_a t_a)'$$

and

$$(u_a t_a)' = u_a t'_a + t_a u'_a.$$

Whence

$$u_a\{B_2(u_a t_a)'\}' = (u_a B_2 u_a t'_a)' + (u_a B_2 u'_a t_a)'$$
$$- u'_a B_2 u_a t'_a - u'_a B_2 u'_a t_a.$$

Now since all the terms in $I_1$ which contain $t_a$ undifferentiated must cancel, we reject the last term and also the couple 2 and 3, because, as $n = 1$, they could be united into one term which would, as $l = 0$, contain $t_a$ undifferentiated. For the same reason the second term in (29) is rejected, and we have

$$I_1 = (B_2 u^2_a t'_a)' = (C_2 t'_a)'$$

and

$$\int I_1 \, dx = C_2 t'_a,$$

where

$$C_2 = B_2 u^2_a. \tag{30}$$

**171.** Resuming (27), $\delta U$ can now be integrated by parts, thus:

$$\delta U = -\frac{1}{2}\int_{x_0}^{x_1} I_1 \, t_a \, dx$$

$$= -\frac{1}{2}\left(C_2 t'_a t_a\right)_1 + \frac{1}{2}\left(C_2 t'_a t_a\right)_0 + \frac{1}{2}\int_{x_0}^{x_1} C_2 t'^2_a \, dx. \tag{31}$$

The following equations will also hold:

$$t = \frac{\delta y}{u}, \qquad t' = \frac{u\delta y' - \delta y u'}{u^2}, \qquad t_a = \frac{t'}{u_a} = \frac{u\delta y' - \delta y u'}{u^2 u_a}.$$

Now since $u$ does not vanish at either limit, and $\delta y$ and $\delta y'$ vanish at both, it is evident, from the above value of $t'$, that $t_1'$ and $t_0'$ will become zero, which will cause $t_a$ to vanish at the limits. Then putting for $u_a$ its value $v'_a$, and for $C_2$ the value obtained by referring to equations (30) (22), and (9), we have

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} a_{y'y'} u^2 v'^2_a \left(\frac{t'}{v'_a}\right)'^2 dx. \tag{32}$$

**172.** We must now determine the form of the quantity $v_a$, and for this purpose we must evidently solve (25). Now by comparing this equation with (12), we see that $v_a$ is what $t$ must become in order to render $\int_{x_0}^{x_1} I dx$ zero; that is, to render $I$ zero. But $I = u\delta M$, which will at once appear if, in the final value of $\delta M$ given in Art. 164, we write $\delta y = ut$, $\delta y' = (ut)'$ and $\delta y'' = (ut)''$, which will in no way restrict the values of the

variations. Hence, since $u$ does not vanish, we must, when $I$ is zero, have $\delta M$ zero. Now we already know that this condition will be satisfied by making

$$\delta y = \frac{dy}{dc_1}\delta c_1 + \frac{dy}{dc_2}\delta c_2 + \frac{dy}{dc_3}\delta c_3 + \frac{dy}{dc_4}\delta c_4$$

$$= ar_1 + br_2 + cr_3 + dr_4, \tag{33}$$

and this condition can, since $A$, $A_1$ and $A_2$ are not in our power, be satisfied in no other way. For the integration of the equation $M = 0$ gives $y$ as a function of $x$ and certain constants, the form of the function being determined, and the values of these constants only being undetermined. Therefore, since $x$ does not receive any variation, any change which cannot be produced in $y$ by varying the constants would cause some change in the form of the function, and hence $y$, when thus changed, could no longer satisfy the equation $M = 0$, which it must do in order that $\delta M$ may vanish. This reasoning is evidently applicable whatever be the order of $M$.

Now it is evident that we can cause the second member of (33), which we know to represent the most general form of $u$, to assume various values for the same value of $x$ by various determinations of the arbitrary constants $a$, $b$, etc. Let $u$ and $v$ be any two such values, so that we may write

$$u = a_1 r_1 + a_2 r_2 + a_3 r_3 + a_4 r_4, \tag{34}$$

$$v = b_1 r_1 + b_2 r_2 + b_3 r_3 + b_4 r_4. \tag{35}$$

But since $\delta y = ut$, if we make $t = \frac{v}{u}$, $\delta y$ will become $v$, and the equation $\delta M = 0$ will be satisfied, as will also the equation $I = 0$. Moreover, this is the only solution; since, by suitably determining the constants in $v$, $\frac{v}{u}$ can be made to equal

any value of $t$ which will render $I$ zero, and therefore every value which will render $\int I\,dx$ zero.

**173.** But the value of $v_a$ is not yet fully determined. For although, by substituting $\frac{v}{u}$ for $t$, we shall render $I$ zero whatever be the system of arbitrary constants employed in $v$, we shall not, by such a substitution, necessarily satisfy (25). Because when $I$ vanishes independently of any particular value of $v$, $\int I\,dx$ is merely a constant. Hence all that we can say is that the relation $v_a = \frac{v}{u}$ will render the second member of (25) a constant. Moreover, it is the only relation which will render it a constant, because it is the only value of $t$ which will cause $I$ to vanish. Hence, since zero is a constant, if any real value of $v_a$ exist, it must be capable of being expressed in the form $\frac{v}{u}$; only the eight constants, $a_1$, etc., $b_1$, etc., must be so related as to satisfy (25).

One of these relations will immediately appear. For, examining (25), we see that it is a differential equation of the third order in $v_a$; and hence by integration we should obtain $v_a$ as a function involving not more than three perfectly arbitrary constants of integration. If, however, we understand only by $u$ and $v$ any two quantities of the form given in (34) and (35) in which the eight constants are so related that $\frac{v}{u}$, when put for $t$, will satisfy (25), which relation must cause the constants to be so combined that $\frac{v}{u}$ may contain not more than three arbitrary constants, then we may write

$$v_a = \frac{v}{u}. \tag{36}$$

**174.** Although this relation between the constants was noticed by Jacobi, many subsequent writers have fallen into the error of supposing that they are entirely independent, and have thus rendered this portion of their explanation untrustworthy. Among these writers is M. Delaunay, who was followed by Prof. Jellett. The latter, on page 95, makes a statement which would with our notation be equivalent to saying that whatever value of $t$ will make $I$ vanish, will also render $\int I\,dx$ zero, which is manifestly untrue.

**175.** We may now proceed to the final transformation of the value of $\delta U$ given in (32). We have, from (36),

$$v'_a = \frac{uv' - vu'}{u^2}, \qquad t = \frac{\delta y}{u}, \qquad t' = \frac{u\delta y' - \delta y u'}{u^2}.$$

Therefore

$$\frac{t'}{v'_a} = \frac{u\delta y' - \delta y u'}{uv' - vu'};$$

and

$$\left(\frac{t'}{v'_a}\right)' = \frac{(uv' - vu')(u\delta y' - \delta y u')' - (u\delta y' - \delta y u')(uv' - vu')'}{(uv' - vu')^2}.$$

But

$$(uv' - vu')' = uv'' - vu'',$$

and

$$(u\delta y' - \delta y u')' = u\delta y'' - \delta y u''.$$

Substituting these values in (32), reducing, and factoring with reference to $\delta y$, $\delta y'$ and $\delta y''$, we finally obtain

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} a_{y'y'}$$

$$\left\{\frac{(u'v'' - v'u'')\,\delta y - (uv'' - vu'')\,\delta y' + (uv' - vu')\,\delta y''}{uv' - vu'}\right\}^2 dx. \quad (37)$$

From this equation we see that to render $U$ a maximum or a minimum, $a_{y'y'}$ must be of invariable sign, and should also remain finite throughout the range of integration, and not vanish permanently. If these conditions be fulfilled, it is necessary also that the second factor of (37) should not permanently vanish, and it ought also to remain always finite. The first condition will always be satisfied. For if in any case it were not, we would have

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \delta M \delta y \, dx = 0; \tag{38}$$

and since every element of this integral must have the same sign as $a_{y'y'}$, which is invariable, (38) can only be satisfied by making $\delta M \ \delta y$ constant. But since $\delta y_1$ and $\delta y_0$ are zero, this constant must be zero also, which would render it necessary that $\delta M$ should vanish. But this, as we have shown, would not happen unless $u$ or $ku$ be an admissible value of $\delta y$; and since, as explained in Art. 165, we assume in (37) that such is not the case, it is evident that the factor in question cannot permanently vanish.

Hence we see that if $a_{y'y'}$ be of invariable sign, while $\delta U$ cannot be made to vanish by the use of $u$ or $ku$, as indicated in Art. 165, neither can it be made to vanish by any other mode of varying $y$, $y'$ and $y''$. To satisfy the second condition it is necessary that the denominator in (37) shall not vanish, and that the coefficients of $\delta y$ and $\delta y'$ in the numerator shall both remain finite. That is, we must be able to so determine the constants that $uv' - vu'$ may not vanish, while $u$, $u'$, $u''$, $v$, $v'$ and $v''$ must at the same time remain finite. But before we can examine these conditions, we must be able to express these coefficients of $\delta y$, $\delta y'$ and $\delta y''$ as functions of $x$, and perfectly arbitrary constants, and we shall next consider how this may be effected.

**176.** Now from (34) and (35) we have

$$
\left.
\begin{aligned}
u &= a_1 r_1 + a_2 r_2 + a_3 r_3 + a_4 r_4, \\
u' &= a_1 r_1' + a_2 r_2' + a_3 r_3' + a_4 r_4', \\
u'' &= a_1 r_1'' + a_2 r_2'' + a_3 r_3'' + a_4 r_4''. \\
v &= b_1 r_1 + b_2 r_2 + b_3 r_3 + b_4 r_4, \\
v' &= b_1 r_1' + b_2 r_2' + b_3 r_3' + b_4 r_4', \\
v'' &= b_1 r_1'' + b_2 r_2'' + b_3 r_3'' + b_4 r_4''.
\end{aligned}
\right\} \tag{39}
$$

As we wish to substitute these quantities in the various parts of the second member of (37), we can avoid tedious multiplications and exhibit the results more explicitly by the use of determinants. For (37) may evidently be written

$$
\begin{aligned}
\delta U &= \frac{1}{2}\int_{x_0}^{x_1} a_{y''y''} \frac{L^2}{L^2_{y''}}\, dx \\
&= \frac{1}{2}\int_{x_0}^{x_1} a_{y''y''} \left(\frac{L_y \delta y - L_{y'} \delta y' + L_{y''} \delta y''}{L_{y''}}\right)^2 dx,
\end{aligned} \tag{40}
$$

where

$$
\left.
\begin{aligned}
L &= \begin{vmatrix} \delta y, & \delta y', & \delta y'' \\ u, & u', & u'' \\ v, & v', & v'' \end{vmatrix}, \\
L_{y''} = \begin{vmatrix} u, & u' \\ v, & v' \end{vmatrix}, \quad
L_y &= \begin{vmatrix} u', & u'' \\ v', & v'' \end{vmatrix}, \quad
L_{y'} = \begin{vmatrix} u, & u'' \\ v, & v'' \end{vmatrix}.
\end{aligned}
\right\} \tag{41}
$$

Now for convenience we shall denote any determinant of the second order containing two $a$'s and two $b$'s by the numerical suffixes of its first element, and similarly determinants with respect to $r$, $r'$, etc., will be denoted by the numerical suffixes

of their first elements, together with the accents of $r$. Then, since $u'$, $u''$, $v'$ and $v''$ have the forms given in (39), while $L_y$ is a determinant of these quantities, we can, by a well-known principle of the subject, at once exhibit $L_y$ thus:

$$L_y = 12 \cdot 1'2'' + 13 \cdot 1'3'' + 14 \cdot 1'4'' \\ + 23 \cdot 2'3'' + 24 \cdot 2'4'' + 34 \cdot 3'4''; \quad (42)$$

and in like manner we obtain

$$\left. \begin{array}{l} L_{y'} = 12 \cdot 12'' + 13 \cdot 13'' + 14 \cdot 14'' + 23 \cdot 23'' + 24 \cdot 24'' + 34 \cdot 34'', \\ L_{y''} = 12 \cdot 12' + 13 \cdot 13' + 14 \cdot 14' + 23 \cdot 23' + 24 \cdot 24' + 34 \cdot 34'. \end{array} \right\} \quad (43)$$

Hence if we regard the determinants 12, 13, 14, 23, 24, and 34 as new constants, we see that the eight constants in $u$ and $v$ have so combined as to leave but six in equation (40). If now we divide $L_y$, $L_{y'}$ and $L_{y''}$ by one of these constants, as 12, and denote the respective quotients by $M_y$, $M_{y'}$ and $M_{y''}$, we may, without altering the value of equation (40), substitute these quantities for $L_y$, $L_{y'}$ and $L_{y''}$. Hence we require only to determine the forms of these quantities. But if we write

$$a = \frac{13}{12}, \quad b = \frac{14}{12}, \quad c = \frac{23}{12}, \quad d = \frac{24}{12}, \quad e = \frac{34}{12}, \quad (44)$$

then

$$\left. \begin{array}{l} M_y = 1'2'' + a\,1'3'' + b\,1'4'' + c\,2'3'' + d\,2'4'' + e\,3'4'', \\ M_{y'} = 12'' + a\,13'' + b\,14'' + c\,23'' + d\,24'' + e\,34'', \\ M_{y''} = 12' + a\,13' + b\,14' + c\,23' + d\,24' + e\,34'. \end{array} \right\} \quad (45)$$

We have now but five constants to consider, and the last of these may be expressed in terms of the other four. For we have

$$12 \cdot 34 + 23 \cdot 14 - 13 \cdot 24 = 0, \quad (46)$$

an equation which will be found upon trial to be identically true. Hence

$$\frac{34}{12}+\frac{23}{12}\cdot\frac{14}{12}-\frac{13}{12}\cdot\frac{24}{12}=e+cb-ad=0$$

and

$$e=ad-bc=\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}, \tag{47}$$

which value being substituted in (45) will give $M_y$, $M_{y'}$ and $M_{y''}$ as functions of four constants only.

Our reasoning thus far would hold even were the eight constants which enter $u$ and $v$ entirely unrestricted. But since these constants must have such mutual relations as will satisfy equation (25), where we now know that $v_a$ is put for $\frac{v}{u}$, the four remaining constants must also be subject to some restriction, or conditioning equation, which will enable us to express $M_y$, $M_{y'}$ and $M_{y''}$ as functions of not more than three perfectly arbitrary constants. But to determine this last relation in any particular case it will be convenient to present equation (25) under another form, and this we now proceed to do.

**177.** Assume the equations

$$Az+(A_1 z')'+(A_2 z'')''=f$$

and

$$Au+(A_1 u')'+(A_2 u'')''=F.$$

Then

$$uf-zF=uAz-zAu+u(A_1 z')'-z(A_1 u')'$$
$$+u(A_2 z'')''-z(A_2 u'')''=i+k. \tag{48}$$

Now

$$u(A_1 z')' = (uA_1 z')' - u'A_1 z'$$

and

$$z(A_1 u')' = (zA_1 u')' - z'A_1 u'.$$

Whence

$$i = A_1(uz' - zu')'. \tag{49}$$

Also,

$$u(A_2 z'')'' = (uA_2 z'')'' - 2(u'A_2 z'')' + u''A_2 z'';$$

and developing the remaining term in like manner, and subtracting, we have

$$k = \{A_2(uz'' - zu'')\}'' - 2\{A_2(u'z'' - z'u'')\}'. \tag{50}$$

But since the second members of (49) and (50) are integrable once, if we add these equations, obtaining thereby the value of $uf - zF$, and then integrate, we shall have

$$\int \{uf - zF\} dx = A_1(uz' - zu') - 2A_2(u'z'' - z'u'')$$
$$+ \{A_2(uz'' - zu'')\}'. \tag{51}$$

Now put $\delta y$ for $z$, and let $u$ be such a value of $z$ or $\delta y$ as will render $F$ zero. Then the second term will disappear from the first member of (51), and the remaining term will become $\int I dx$; and we shall have

$$\int I dx = A_1(u\delta y' - \delta y u') - 2A_2(u'\delta y'' - \delta y' u'')$$
$$+ \{A_2(u\delta y'' - \delta y u'')\}' = B_1 t' + (B_2 t'')'. \tag{52}$$

But since $t = \dfrac{\delta y}{u}$ and $v_a = \dfrac{v}{u}$, we have only to change $\delta y$ into

$v$ in order to cause $t$ to become $v_a$. Hence, finally, (25) may be written

$$A_1(uv' - vu') - 2A_2(u'v'' - v'u'') + \{A_2(uv'' - vu'')\}' = 0; \quad (53)$$

and as we may divide by any constant, we may write, as the final conditioning equation,

$$A_1 M_{y''} - 2A_2 M_y + (A_2 M_{y'})' = 0. \quad (54)$$

It also appears by differentiation that

$$L_{y'} = L'_{y''} \text{ and } L'_{y'} = L_y + uv''' - vu''' = L_y + L_x,$$

where $L_x$ is exhibited by determinants in the same manner as the other $L$'s. Hence, dividing these equations, as before, by the determinant constant 12, we have

$$\left.\begin{aligned} M_{y'} = M'_{y''}, \qquad M'_{y'} = M_y + M_x, \\ M_x = 12''' + a\, 13''' + b\, 14''' + c\, 23''' + d\, 24''' + e\, 34'''. \end{aligned}\right\} \quad (55)$$

It is evident, however, that in order to apply equations (54) and (55) to the reduction of the constants, we must determine the particular forms which are assumed by $A_1$, $A_2$, $r_1$, $r_2$, $r_3$ and $r_4$, which cannot be done so long as the problem remains wholly general.

**178.** The following example is presented merely as a means of illustrating the preceding discussion.

### Problem XXXI.

*It is required to apply Jacobi's Theorem to Prob. V.*

Here $a_{y'y'} = 2$, so that we have next to consider whether the terms of the second order can be made to vanish by the

use of $u$ or $ku$. Now the general solution, equation (6), Art. 42, may be written

$$y = x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4. \tag{1}$$

Hence we have the following equations:

$$\left.\begin{array}{llll}
r_1 = x^3, & r_2 = x^2, & r_3 = x, & r_4 = 1, \\
r_1' = 3x^2, & r_2' = 2x, & r_3' = 1, & r_4' = 0, \\
r_1'' = 6x, & r_2'' = 2, & r_3'' = 0, & r_4'' = 0, \\
r_1''' = 6, & r_2''' = 0, & r_3''' = 0, & r_4''' = 0.
\end{array}\right\} \tag{2}$$

$$\left.\begin{array}{l}
u = a_1 x^3 + a_2 x^2 + a_3 x + a_4, \\
u' = 3a_1 x^2 + 2a_2 x + a_3.
\end{array}\right\} \tag{3}$$

Now if the constants in $u$ can be so taken that $u$ and $u'$ shall vanish twice or more, simultaneously, within the limits of integration, the terms of the second order can be made to vanish by the use of $u$, and we have in general neither a maximum nor a minimum.

Now if $u$ and $u'$ can satisfy these conditions, let $x_2$ and $x_3$ be two values of $x$ for which they vanish simultaneously. Then we must have

$$x_2^3 + \frac{a_2}{a_1} x_2^2 + \frac{a_3}{a_1} x_2 + \frac{a_4}{a_1} = 0, \tag{4}$$

$$x_3^3 + \frac{a_2}{a_1} x_3^2 + \frac{a_3}{a_1} x_3 + \frac{a_4}{a_1} = 0, \tag{5}$$

$$x_2^2 + \frac{2a_2}{3a_1} x_2 + \frac{a_3}{3a_1} = 0, \tag{6}$$

$$x_3^2 + \frac{2a_2}{3a_1} x_3 + \frac{a_3}{3a_1} = 0. \tag{7}$$

Subtracting (7) from (6), and (5) from (4), and dividing by $x_2 - x_3$, we have

$$x_2 + x_3 + \frac{2a_2}{3a_1} = 0, \tag{8}$$

$$x_2^2 + x_2 x_3 + x_3^2 + \frac{a_2}{a_1}(x_2 + x_3) + \frac{a_3}{a_1} = 0. \tag{9}$$

Substituting in (9) the value of $\frac{a_2}{a_1}$ from (8), we have

$$x_2^2 + x_2 x_3 + x_3^2 - \frac{3}{2}(x_2 + x_3)(x_2 + x_3) + \frac{a_3}{a_1} = 0$$

$$= -\frac{x_2^2}{2} - 2x_2 x_3 - \frac{x_3^2}{2} + \frac{a_3}{a_1}. \tag{10}$$

Substituting in (6) the values of $\frac{2a_2}{3a_1}$ and $\frac{a_3}{3a_1}$ from (8) and (10), we have, after reducing,

$$x_2^2 - 2x_2 x_3 + x_3^2 = 0.$$

Hence $x_2$ and $x_3$ cannot be different values of $x$, and the terms of the second order cannot be made to vanish by the use of $u$. But since, as we have seen in Art. 175, these terms can be made to vanish by no other mode of varying $y$, we are sure of a minimum, unless, indeed, we cannot prevent $M_y$ or $M_{y'}$ from becoming infinite, or $M_{y''}$ from vanishing within the range of integration; and these points we shall next consider.

**179.** Finding, by the use of equations (2), the values of $M_{y''}$, $M_{y'}$ and $M_y$ in equations (45), Art. 176, and also that of $M_x$ in equations (55), Art. 177, we shall obtain

$$\left.\begin{aligned} M_{y''} &= -x^4 - 2ax^3 - 3bx^2 - cx^2 - 2dx - e, \\ M_{y'} &= -4x^3 - 6ax^2 - 6bx - 2cx - 2d, \\ M_y &= -6x^2 - 6x - 2c, \\ M_x &= -6x^2 - 6ax - 6b. \end{aligned}\right\} \tag{11}$$

Now since $a_{y''y''} = 2$, we see from equations (9), Art. 164, that $A_1 = 0$ and $A_2 = 2$. Hence, in this case, equation (54), Art. 177, becomes

$$-4M_y + 2M'_{y'} = 0 = -2M_y + 2M_x,$$

as will appear from equations (55) of the same article. Equating the values of $M_y$ and $M_x$, we have $c = 3b$. Now taking the value of $e$ from equation (47), Art. 176, and then substituting in the first of equations (11) $3b$ for $c$, we shall have, after changing signs,

$$-M_{y''} = x^4 + 2ax^3 + 6bx^2 + 2dx + ad - 3b^2, \qquad (12)$$

$$-M_{y'} = 4x^3 + 6ax^2 + 12bx + 2d, \qquad (13)$$

$$-M_y = 6x^2 + 6ax + 6b. \qquad (14)$$

It therefore at once appears that neither $M_{y'}$ nor $M_y$ can become infinite so long as $a$, $b$, $d$ and $x$ remain finite. We can also evidently choose these constants in such a manner that $M_{y''}$ shall not vanish within the limits of integration. For suppose, for example, that we make both $a$ and $d$ zero. Then to render the equation

$$\frac{M_{y''}}{3} = b^2 - 2x^2b - \frac{x^4}{2} = 0$$

possible, we must have

$$b = x^2 \pm \frac{2x^2}{\sqrt{3}}.$$

Hence if we assume $b$ greater or less than this value can become within the limits of integration, and also make $a$ and $d$ zero, we shall secure that $M_{y''}$ will not vanish at all as we pass from $x_0$ to $x_1$; and therefore, as all the requisite conditions can be satisfied, we are in this case sure of a minimum.

**180.** We have, then, the following general method of applying the theorem of Jacobi in this case.

First find whether $a_{y'y'}$ remains finite, does not vanish permanently, and is of invariable sign throughout the range of integration; because if these conditions be not fulfilled there is no need of any further investigation. But if they be satisfied, next try whether $\delta U$ can be made to vanish by the use of $u$.

For this purpose we write

$$u = r_1 + \frac{a_2}{a_1} r_2 + \frac{a_3}{a_1} r_3 + \frac{a_4}{a_1} r_4$$

and

$$u' = r_1' + \frac{a_2}{a_1} r_2' + \frac{a_3}{a_1} r_3' + \frac{a_4}{a_1} r_4'.$$

Then if $\delta U$ can be made to vanish by the use of $u$, the following equations must be possible:

$$u_2 = 0, \qquad u_2' = 0, \qquad u_3 = 0, \qquad u_3' = 0,$$

where neither $x_2$ nor $x_3$ must fall without the limits of integration. To determine the possibility of these equations we first eliminate between them the constants $\frac{a_2}{a_1}$, $\frac{a_3}{a_1}$ and $\frac{a_4}{a_1}$, by which we shall arrive at an equation containing only $x_2$, $x_3$, and such constants as enter $y$ in the equation of the curve represented by the solution. It may then happen, as in the preceding example, that we can determine the possibility of satisfying this equation within the limits of integration. Or, if necessary, we can, by using the values of $y, y'$, etc., obtained from the equation of the curve, eliminate all constants but numbers, thus securing a numerical equation between $x_2$, $x_3$, $y_2$, $y_3$, $y_2'$, etc., which it must be possible to satisfy within the limits of integration.

If, then, it be possible to satisfy this equation, we infer, as in Case 1, that we have neither a maximum nor a minimum. But if the limiting values of $u$ and $u'$ cannot be made to vanish simultaneously, we may assume that we have a maximum or a minimum according as $a_{y''y''}$ is negative or positive.

This assumption will, however, be subject to any doubt arising from the possibility that we may not be able by any selection of constants to prevent $M_y$ or $M_{y'}$ from becoming infinite, or $M_{y''}$ from vanishing for some value of $x$ within the limits of integration, thus rendering the corresponding element of $\delta U$ infinite. To dispose of this doubt, we must, in the next place, actually find the quantities $M_y$, $M_{y'}$ and $M_{y''}$, and possibly $M_x$, as functions of $x$, and but three arbitrary constants, any constants which may enter $r_1$, etc., not being reckoned. But this latter step, which will usually involve difficulty, may in general be omitted.

**181.** Some exceptions also occur in the treatment of this case which are similar to those mentioned under Case 1 (see Art. 157). We shall, however, merely indicate these exceptions, the discovery of which appears to be due likewise to Spitzer. (See Todhunter's History of Variations, Art. 276.)

Suppose, first, $a_{y''y''}$ to become zero. Then it is shown that in order that $U$ may become a maximum or a minimum, $A_1$ must have respectively a positive or negative sign throughout the range of integration.

Suppose, in the second place, that we have $a_{y''y''}$ zero, and also $A_1$ zero, $A$ and $A_1$ having the values given in equations (9), Art. 164. Then it is shown that in order that $U$ may become a maximum or a minimum, $A$ must be respectively negative or positive throughout the range of integration. Moreover, in this case, as in Case 1, Art. 157, we shall find that the equation $M = 0$ will not be a differential equation in $y$, but merely an ordinary algebraic equation, and that therefore $y$ will, without integration, be determined as a function of

$x$. Hence, geometrically, there will be no solution unless the limiting values of $y$ and $y'$ happen to satisfy the equation of a particular curve or class of curves.

Suppose, lastly, that $a_{y'y'}$, $A_1$ and $A$ become severally zero. Then, as in Case 2, Art. 157, the equation $U=\int_{x_0}^{x_1} V dx$ is capable of being integrated, and therefore the maximum or minimum state of $U$ must, if at all, be found by the differential calculus; and if the limiting values of $x$, $y$ and $y'$ be fixed, $U$ will have neither a maximum nor a minimum state.

It is evident that in all these cases $V$ contains merely the first power of $y''$, and they are, therefore, like those in Art. 157, only examples of Exception 2, Art. 51.

**182.** As the most general case of Jacobi's Theorem is precisely analogous to that already explained, and as it is rather of analytical than practical importance, we shall merely indicate the method of effecting the required transformation.

CASE 3.

Let $U=\int_{x_0}^{x_1} V dx$, where $V$ is any function of $x, y, y', \ldots . y^{(n)}$. Then the general solution $M=0$ will usually give $y$ as a function of $x$ and $2n$ arbitrary constants of integration, and these $2n$ arbitrary constants must be so determined as to satisfy the conditions at the limits, where we shall always suppose the limiting values of $x, y, y', \ldots . y^{(n-1)}$ to be assigned.

Now, as before, since these conditions hold at the limits, and the terms of the first order must vanish, we may write

$$\delta U=\frac{1}{2}\int_{x_0}^{x_1} \delta M \, \delta y \, dx,$$

and may then, by Lemma I., put $\delta M$ under the form

$$\delta M = A\delta y + \left(A_1 \delta y'\right)' + \text{etc.} \ldots . + \left(A_n \delta y^{(n)}\right)^{(n)}.$$

We shall also have, in this case,

$$u = A_1 r_1 + A_2 r_2 + \text{etc.} \ldots + A_{2n} r_{2n}.$$

Hence, by changing $\delta y$ into $ut$, and integrating by parts with the aid of Note to Lemma II., we shall obtain a result which may be written

$$\delta U =$$

$$-\frac{1}{2}\int_{x_0}^{x_1} \left\{ B_1 t' + \left(B_2 t''\right)' + \text{etc.} \ldots + \left(B_n t^{(n)}\right)^{(n-1)} \right\} t' dx.$$

Then, as formerly, putting $u_a t_a$ for $t'$, and integrating again by parts, we have

$$\delta U =$$

$$\frac{1}{2}\int_{x_0}^{x_1} \left\{ C_2 t'_a + \left(C_3 t''_a\right)' + \text{etc.} \ldots + \left(C_n t_a{}^{(n-1)}\right)^{(n-2)} \right\} t'_a dx.$$

In this equation we may change $t'_a$ into $u_b t_b$, where $u_b = w'_b$, and $w_b$ is a quantity which satisfies the differential equation

$$C_2 w'_b + \left(C_3 w''_b\right)' + \text{etc.} \ldots + \left(C_n w_b{}^{(n-1)}\right)^{(n-2)} = 0.$$

Making this change, and integrating by parts, as before, we have

$$\delta U = -\frac{1}{2}\int_{x_0}^{x_1} \left\{ D_3 t'_b + \left(D_4 t''_b\right)' + \text{etc.} \ldots \right.$$

$$\left. + \left(D_n t_b{}^{(n-2)}\right)^{(n-3)} \right\} t'_b dx.$$

Continuing this process $n$ times, we shall evidently arrive at a result which may be written

$$\delta U = \pm \frac{1}{2}\int_{x_0}^{x_1} H T^2 dx,$$

the positive or negative sign being used according as $n$ is even or odd.

Now it is evident, from the mode in which the integration is effected, that $H$ must equal $A_n u^2 u^2_a u^2_b \ldots$, and $A_n = \pm \frac{d^2 V}{dy^{(n)2}}$, the positive or negative sign being used according as $n$ is even or odd, as will at once appear if we form the functions $A$, $A_1$, etc., by Note to Lemma I.

**183.** Let us next consider the quantities $u$, $u_a$, $u_b$, etc. We have, by the same reasoning as that hitherto employed,

$$\left.\begin{array}{ll} u = a_1 r_1 + a_2 r_2 + \text{etc.} \ldots + a_{2n} r_{2n}, & u_a = v'_a, \\ v_a = \dfrac{v}{u}, \quad v = b_1 r_1 + b_2 r_2 + \text{etc.} \ldots + b_{2n} r_{2n}. & \end{array}\right\} \quad (1)$$

But the $2n$ constants $a$ and the $2n$ constants $b$ are not entirely independent, but must be so related that $v_a$ may satisfy the equation

$$B_1 v'_a + \left(B_2 v''_a\right)' + \text{etc.} \ldots + \left(B_n v_a^{(n)}\right)^{(n-1)} = 0; \quad (2)$$

that is, $v_a$, when put for $t$, must render $\int I dx$ zero.

The following relations are also evidently true:

$$I = u\delta M, \qquad I_1 = u_a \int I dx, \qquad I_2 = u_b \int I_1 dx, \text{ etc.} \quad (3)$$

Now to determine the nature of $u_b$, we see from (1) that $w_b$ is a quantity which, being put for $t_a$, will render $\int I_1 dx$ zero; that is, will render $I_1$ or $u_a \int I dx$ zero, will render $\int I dx$ or $u\delta M$ zero, will render $\delta M$ zero. But since, in $\int I dx$, $t'$ is replaced by $u_a t_a$,

$t_a = \frac{t'}{u_a}$; and in order to render that integral zero, the $t'$ in the value of $t_a$ just given must now be so restricted as to satisfy the equation

$$B_1 t' + \left(B_2 t''\right)' + \text{etc.} \ldots . + \left(B_n t^{(n)}\right)^{(n-1)} = 0;$$

that is, it must render $\int I dx$ zero, or $I$ zero, or $u\delta M$ zero, or $\delta M$ zero. But always $t = \frac{\delta y}{u}$, and we can make $\delta M$ vanish only by making $\delta y$ equal to some quantity of the general form of $u$.

Assume $\delta y = w$, where

$$w = c_1 r_1 + c_2 r_2 + \text{etc.} \ldots . + c_{2n} r_{2n}.$$

This will make $\delta M$ vanish; and if the $2n$ constants $c$ and the $2n$ constants $a$ be suitably connected, $t$, which now equals $\frac{w}{u}$ or $w_a$, will also satisfy the equation $\int I dx = 0$, which would not necessarily happen if these constants were entirely independent.

We have now, as the value of $t_a$ which was required to render $\int I dx$ zero,

$$t_a = \frac{\left(\frac{w}{u}\right)'}{\left(\frac{v}{u}\right)'}.$$

But it does not follow that every value of $t_a$ which will render $\int I dx$ zero will also render $\int I_1 dx$ zero, and $w_b$ must be such a value of $t_a$. Still it is evident that $w_b$ can be of no other

general form than that just given for $t_a$; only, in addition to the relations already noticed between the constants, the constants in $u$, $v$ and $w$ must be so related that $\frac{\left(\frac{w}{u}\right)'}{\left(\frac{v}{u}\right)}$, may render $\int I_1\,dx$ zero.

In a similar manner we may determine $u_c = z'_c$, but will then be obliged ultimately to introduce into $\delta U$ another quantity of the form

$$z = d_1 r_1 + d_2 r_2 + \text{etc.} \ldots + d_{2n} r_{2n}.$$

Moreover, these four sets of constants will then be subjected to three more conditions; six in all. For $z_c$ must be so taken as to reduce to zero the following expressions:

$$\int I_2\,dx, \qquad \int I_1\,dx, \qquad \int I\,dx, \qquad \delta M;$$

the last condition serving merely to introduce $z$, but imposing no restriction upon its constants.

Thus it appears that each increase by unity of $n$ will introduce into $\delta U$ one more quantity like $z$, and that each such new quantity will require one more additional condition than did its predecessor, the first condition being introduced by the second of these quantities.

Now $T$ can always be found in terms of the preceding quantities. For we have

$$t = \frac{\delta y}{u}, \qquad t_a = \frac{t'}{u_a}, \qquad t_b = \frac{t'_a}{u_b}, \qquad \text{etc.} \tag{4}$$

Whence we see that by means of $T$ the final value of $\delta U$ will be made to involve $\delta y$, $\delta y'$, .... $\delta y^{(n)}$, which should evidently be the case.

**184.** The analogy of the preceding cases would lead us to expect that when the reductions indicated in the last article are performed, $\delta U$ will assume a determinant form; and such is the fact. This subject, and indeed the whole theorem of Jacobi, has been most elaborately discussed by Otto Hesse in a paper which may be found in the 54th volume of Crelle's *Mathematical Journal* for 1857, p. 227, and we are indebted to this author for much of the preceding discussion, and in particular for that part which exhibits the relation between the constants and the manner in which they combine and reduce. We shall, however, here merely give some of his results.

Let $u, v, w, \ldots . X$ be $n$ quantities which, being put for $\delta y$, will severally render $\delta M$ zero. Then we see from Art. 183 that $\delta U$ will involve all these quantities. Let

$$L = \begin{vmatrix} \delta y, & \delta y', & \ldots . & \delta y^{(n)} \\ u, & u', & \ldots . & u^{(n)} \\ v, & v', & \ldots . & v^{(n)} \\ . & . & . & . \\ . & . & . & . \\ X, & X', & \ldots . & X^{(n)} \end{vmatrix}, \qquad L_{y^n} = \begin{vmatrix} u, & u', & \ldots . & u^{(n-1)} \\ v, & v', & \ldots . & v^{(n-1)} \\ . & . & . & . \\ . & . & . & . \\ X, & X', & \ldots . & X^{(n-1)} \end{vmatrix};$$

$L$ being a determinant of the order $n + 1$, and $L_{y^n}$ a determinant of the order $n$. Then Hesse shows that $\delta U$ will take the form

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \frac{d^2 V}{dy^{(n)2}} \left(\frac{L}{L_{y^n}}\right)^2 dx.$$

It is also evident, from Art. 183, that the number of the conditioning equations between the constants involved in $u, v, \ldots . X$ must be the sum $1 + 2 + 3 +$ etc. $\ldots . + n - 1$, or $\frac{n(n-1)}{2}$.

We may here collect a few of these conditioning equations, the first arising from $u_a$, the next two from $u_b$, and the last three from $u_c$.

$$B_1 v'_a + \text{etc.} \dots + \left(B_n v_a^{(n)}\right)^{(n-1)} = 0,$$

$$B_1 w'_a + \text{etc.} \dots + \left(B_n w_a^{(n)}\right)^{(n-1)} = 0,$$

$$C_2 w'_b + \text{etc.} \dots + \left(C_n w_b^{(n-1)}\right)^{(n-2)} = 0,$$

$$B_1 z'_a + \text{etc.} \dots + \left(B_n z_a^{(n)}\right)^{(n-1)} = 0,$$

$$C_2 z'_b + \text{etc.} \dots + \left(C_n z_b^{(n-1)}\right)^{(n-2)} = 0,$$

$$D_2 z'_c + \text{etc.} \dots + \left(D_n z_c^{(n-2)}\right)^{(n-3)} = 0,$$

where

$$w_a = \frac{w}{u}, \qquad z_a = \frac{z}{u}, \qquad z_b = \frac{z'_a}{v'_a}.$$

Some of these relations are more explicitly exhibited by Hesse, but, for a reason which will presently appear, it is unnecessary to go any further into this matter.

Now

$$z_c = \frac{\left(\frac{z'_a}{v'_a}\right)'}{w'_b}, \qquad w_b = \frac{w'_a}{v'_a}, \qquad \text{and} \quad v_a = \frac{v}{u}.$$

Hence it appears that $w_b$ is a differential expression of the first order, $z_c$ of the second, etc.

**185.** The manner in which the constants enter $\delta U$ is similar to that in Case 2. For $L$ may be written

$$L_y \delta y' + L_{y'} \delta y + \text{etc.} \dots + L_{y^n} \delta y^{(n)},$$

where $L_y$, $L_{y'}$, etc., are themselves determinants of the $n$th order. But if in any of these determinants we substitute the values of its constituents, we know that the determinant will become the sum of products of pairs of determinants, each product consisting of a determinant of the $n$th order in constants, multiplied by a determinant of the same order in the $r$'s and their differential coefficients, there being as many such products as there are combinations of $2n$ numbers, taken $n$ in a set, no two determinants, whether variable or constant, being the same.

This is, however, as far as we can go. For to show, in general, how these determinant constants combine so that $L_y, L_{y'}, \ldots . L_{y^n}$ may be expressed as functions of $x$ and entirely independent constants, is a problem which has not yet been solved. Now in order that no element of $\delta U$ may become infinite, we must be able to so determine the arbitrary constants that $L_{y^n}$ shall not vanish, and that none of the quantities $L_y, L_{y'}, \ldots . L_{y^n}$ may become infinite within the range of integration. But the above defect will prevent us from determining whether or not these conditions can be fulfilled, since it will prevent us from obtaining these quantities as explicit functions of $x$ and entirely independent constants.

**186.** After a general discussion, Hesse considers successively the cases in which $n$ is 1, 2 and 3. In the latter case, the constants will enter $L_y, L_{y'}, \ldots . L_{y^n}$, in the form of twenty determinant constants of the third order, and the conditioning equations will be three in number. Moreover, between these twenty determinants there subsist thirty identical equations analogous to equation (46), Art. 176. Now by division, as before, $L_y$, $L_{y'}, \ldots . L_{y^n}$ become $M_y, M_{y'}, \ldots . M_{y^n}$, and these constants may be reduced to nineteen. Then the three conditioning equations should enable us to reduce them to sixteen, and finally the thirty identical equations are of such a character as to enable us to eliminate but ten more determinants.

Thus it will appear that there remain not more than six irreducible constants. Hesse does not say that these constants are yet perfectly independent, and the author is not prepared to say more than that they appear to be so. For a further discussion of this subject the reader is referred to the paper in question.

**187.** We see then, in general, that in order that $U$ may have a maximum or a minimum state, it is, in the first place, necessary that $\frac{d^2 V}{dy^{(n)2}}$ or $a_{y^n y^n}$ shall remain finite, not vanish permanently, and be of invariable sign throughout the limits which we wish to consider. This principle, however, is not due to Jacobi, it having been enunciated by Legendre as early as the year 1786. Still, the method of discriminating maxima and minima given by Legendre and Lagrange was defective, because it gave no means of determining whether some element of $\delta U$ might not become infinite, as it always employed certain functions which could not be determined. (See Todhunter's History of Variations, Arts. 5, 199.)

If the conditions with regard to $a_{y^n y^n}$ indicate a maximum or a minimum, we must, in the next place, determine whether $\delta U$ can be made to vanish by the use of $u$, since, if it can, there is no need of Jacobi's transformation, and we infer at once that $U$ has neither a maximum nor a minimum state. To make $\delta U$ thus vanish we must be able to satisfy the equations

$$u_2 = 0, \qquad u_2' = 0, \text{ etc.,} \qquad u_2^{(n-1)} = 0,$$

$$u_3 = 0, \qquad u_3' = 0, \text{ etc.,} \qquad u_3^{(n-1)} = 0,$$

where neither $x_2$ nor $x_3$ must fall without the limits of integration.

Geometrically, we may regard $y$ in any proposed solution as the ordinate of a curve whose extremities are the points $x_0, y_0$ and $x_1, y_1$. Then the proposed value of $y$ will render $U$

neither a maximum nor a minimum, if it be possible, by making infinitesimal changes in $y, y', \ldots . y^{(n)}$, to draw another curve meeting the first at the points $x_1, y_1$ and $x_2, y_2$, and having at these points the same values of $y', \ldots . y^{(n-1)}$, and also satisfying the equation $M = 0$.

Now although, when the limiting values of $x$, $y$, $y'$, .... $y^{(n-1)}$ are assigned, all the constants which enter the equation of the curve which satisfies all the conditions of the question are determined, yet as this determination is not alway absolute, allowing us a choice of two or more values, there will in general be more than one such curve, as in Prob. VII., where two catenaries can often be drawn, both satisfying the conditions of the question. Now if such limits be found, in passing along one of these curves, as to render it and another curve coincident between these limits—that is, if the equation of this curve have one or more pairs of equal roots—$\delta U$ to the second order can be made to vanish, and we infer that $U$ has neither a maximum nor a minimum state.

If we can assure ourselves that $\delta U$ will not vanish, then we must, in the third place, determine $L_y, L_{y'}, \ldots . L_{y^{n}}$, in order to ascertain whether or not all the elements of $\delta U$ remain finite. But this point has been already fully treated, and we have seen that this determination cannot always be effected. When this is the case, the theorem of Jacobi is practically subject to the same defect as existed in the method of Legendre and Lagrange. It will appear, however, that by determining the function $u$, which is always possible when the complete integral of the equation $M = 0$ can be obtained, and sometimes when it cannot, we may frequently be able to infer that $U$ has neither a maximum nor a minimum state, even when $a_{y^n y^n}$ is always finite and of invariable sign; and this inference could not be drawn from the above-named method.

**188.** From the cases in which $n$ is 1 and 2, we might naturally expect that some exceptions to the theorem of Jacobi

would arise when $n$ is greater than 2, particularly if $a_{y^n y^n}$ should happen to become zero throughout the range of integration; and such appears to be the fact. For Spitzer has examined also the case in which $n$ is 3, and has shown that certain forms of $V$ give rise to exceptions. We subjoin from Todhunter's History, Art. 278, the following four forms of $V$, which the reader may examine for himself:

$$V=f(x,y,y',y'')+y'''f_a(x,y,y',y''),$$

$$V=f(x,y,y')+y''f_a(x,y,y')+\{f_b(x,y,y',y'')\}',$$

$$V=f(x,y)+y'f_a(x,y)+\{f_b(x,y,y')\}'+\{f_c(x,y,y',y'')\}',$$

$$V=yf(x)+\{f_a(x,y)\}'+\{f_b(x,y,y')\}'+\{f_c(x,y,y',y'')\}',$$

where $f$, $f_a$, $f_b$ and $f_c$ are any functions whatever. Hesse does not mention the existence of any exceptional cases, although he had seen the discussion by Spitzer.

It will be observed that in applying Jacobi's Theorem we have always regarded the limiting values of $x, y, y', \ldots y^{(n-1)}$ as fixed, thus rendering the discussion somewhat restricted. But the solution of the more general problem, that in which these limiting values are also variable, if it be at all possible, has up to the present time baffled the skill of those who have attempted it.

**189.** Before closing this section we must mention one point with regard to the terms of the second order not strictly connected with the theorem of Jacobi.

We have already seen that the simplicity of the form in which these terms appear is often dependent upon our choice of the independent variable, and it may therefore be well to consider particularly how the terms in $\delta U$, derived by regarding $x$ and $y$ successively as the independent variable, are connected, and why they are not identical.

Assume the equation

$$U = \int_{x_0}^{x_1} V dx, \tag{1}$$

where $V$ is any function of $x, y, y', \ldots . y^{(n)}$, the limiting values of $x, y, y', \ldots . y^{(n-1)}$ being fixed. Then, since both $x$ and $y$ are implicitly, at least, involved in $U$, we may regard $y$ as some function of $x$, and may therefore suppose it the ordinate of some primitive curve for the abscissa $x$. Now by varying $U$ we must pass to some derived curve, and let $Y$ or $y + \delta y$ become the ordinate of this curve for the same abscissa $x$.

Next taking $y$ as the independent variable, and expressing $U$ in terms of $y$, $dy$, $x$ and its differential coefficients with respect to $y$, equation (1) may be written

$$U_{,} = \int_{y_0}^{y_1} V_{,} dy, \tag{2}$$

the limiting values of $y$, $x$, $x'$, etc., being also fixed, where $x' = \dfrac{dx}{dy}$, etc. Moreover, $U$ and $U_{,}$ will be identical when the relations between $x$ and $y$ in (1) and (2) are the same; that is, when $y$ is an ordinate of the same primitive curve in both for the same abscissa. Varying (1) and (2) and transforming the terms of the first order, observing that the limiting values are all fixed, we have

$$[\delta U] = \int_{x_0}^{x_1} M \delta y \, dx + \frac{1}{2} \int_{x_0}^{x_1} s \, dx + \int_{x_0}^{x_1} t dx, \tag{3}$$

$$[\delta U_{,}] = \int_{y_0}^{y_1} N \delta x \, dy + \frac{1}{2} \int_{y_0}^{y_1} S \, dy + \int_{y_0}^{y_1} T dy, \tag{4}$$

where brackets denote the entire increment which $U$ and $U_{,}$ receive by variation, the integrals following $M$ denoting respectively the terms of the second order and all those of a higher order, and those following $N$ having a similar significa-

tion. Now supposing $U$ and $U_{\prime}$ identical, the first members of (3) and (4) will become equal if in (2) we so vary $x$ as to obtain the same derived curve as we did from (1) by varying $y$; and this requires $\delta x$ to have such a value that $y$ may be the ordinate of the derived curve for the abscissa $x + \delta x$. Hence, by tracing along the derived curve from the point whose co-ordinates are $x$ and $Y$ to that whose co-ordinates are $x + \delta x$ and $y$, we see that

$$y = Y + Y'\delta x + \frac{1}{2}Y''\delta x^2 + \text{etc.};$$

and putting for $Y$ its value $y + \delta y$, we find

$$\delta y = -y'\delta x + w, \tag{5}$$

$$\delta x = -\frac{\delta y}{y'}, \tag{6}$$

where $w$ contains only terms of an order higher than the first.

Now, since the second members of (3) and (4) are absolutely equal, the terms of the first order in these two members cannot differ by any term of the first order. Hence, and from (6), observing that $dx = \dfrac{dy}{y'}$, we have, to the first order,

$$\int_{x_0}^{x_1} M\,\delta y\,dx = \int_{y_0}^{y_1} N\,\delta x\,dy = \int_{x_0}^{x_1} - N\,\delta y\,dx;$$

so that $N = -M$. Still, denoting by $a$ the terms of the first order, and by $b$ those of the second, in (3), and by $c$ and $d$ the corresponding terms in (4), we cannot say that $a$ and $c$ are absolutely equal, but they cannot differ by more than some term of the second order, and they will in general differ by such a term. In like manner, $a + b$ and $c + d$ cannot differ by any term of the second order, although they may differ by some term of the third, and therefore $b$ and $d$ may, and in general will, differ by a term of the second order.

**190.** Now if $U$ is to be a maximum or a minimum, and we express it successively as in (1) and (2), then $a$ and $c$ must each vanish, because $M$ and $N$ vanish, and we may then find that $d$ is much more simple than $b$; and as these terms must now be equal as far as the second order, because $a$ and $c$ have become zero, we conclude that $b$ must contain an expression which adds nothing of the second order to its value, and that this, by the second method, becomes involved in $c$, thus leaving $b$ in the simpler form $d$.

We know, moreover, that $M$ and $N$ will be entirely independent of the conditions which may be required to hold at the limits, so that the relation $N = -M$ must hold whether the limiting values of $x, y, y'$, etc., be assigned or not. Now if the limiting values of $x$ be fixed, while those of $y$ are variable, then if we change the independent variable to $y$, we may, by regarding the limiting values of $y$ as fixed and those of $x$ as variable, pass to the same derived curve as by the first method. But the abscissæ of the extreme points will now be $x_0 + \delta x_0$ and $x_1 + \delta x_1$, whereas they are required to be $x_0$ and $x_1$ merely. Hence to render $[\delta U_{,}]$ and $[\delta U]$ equal, we must subtract from the former the increment which $U$ would receive in virtue of the change in the limiting values of $x$; and we know that to the first order this increment is

$$V_1 \delta x_1 - V_0 \delta x_0. \tag{7}$$

Now when $[\delta U]$ and $[\delta U_{,}]$ have been made equal, as just explained, it is easy to find what must be the values of the coefficients of $\delta x_1$, $\delta x_0$, $\delta x_1'$, etc. For let $n$ be 2, and $x$ the independent variable. Then, by equation (5), Art. 36, the terms at the upper limit will be

$$(P_1 - Q_1')\delta y_1 + Q_1 \delta y_1'. \tag{8}$$

But from (5) we have, to the first order,

$$\delta y' = -y'' \delta x - y' \frac{d\delta x}{dx};$$

and putting for $dx$ its value $\frac{dy}{y'}$, and observing that $\frac{d\delta x}{dy} = \delta x'$, we have

$$\delta y' = -y''\delta x - y'^2\delta x'. \tag{9}$$

Substituting these values, (8) becomes

$$\{-y_1'(P_1 - Q_1') - y_1''Q_1\}\delta x_1 - y_1'^2 Q_1 \delta x_1', \tag{10}$$

and a similar equation holds for the lower limit.

**191.** Two simple examples will serve to illustrate the preceding discussion.

First assume

$$U = \int_{x_0}^{x_1} y^2 dx = \int_{x_0}^{x_1} V dx, \quad U_{\prime} = \int_{y_0}^{y_1} y^2 x' dy = \int_{y_0}^{y_1} V_{\prime} dy.$$

Whence

$$[\delta U] = \int_{x_0}^{x_1} 2y\, \delta y\, dx + \int_{x_0}^{x_1} \delta y^2 dx,$$

$$[\delta U_{\prime}] = \int_{y_0}^{y_1} y^2\, \delta x\, dy = y_1^2\, \delta x_1 - y_0^2\, \delta x_0 - \int_{y_0}^{y_1} 2y\, \delta x\, dy.$$

Now to render $[\delta U_{\prime}]$ equal to $[\delta U]$, as far as the first order, we must subtract from the former $V_1 \delta x_1 - V_0 \delta x_0$, or $y_1^2 \delta x_1 - y_0^2 \delta x_0$, which will eliminate all the terms at the limits, as it evidently should: and $N = -M$. Still we must recollect that we have made $[\delta U_{\prime}]$ and $[\delta U]$ equal to the first order only.

As a second example, let

$$U = \int_{x_0}^{x_1} y'^2 dx = \int_{x_0}^{x_1} V dx.$$

Then

$$[\delta U] = -2y_1''' \delta y_1 + 2y_1'' \delta y_1' + 2y_0''' \delta y_0 - 2y_0'' \delta y_0'$$

$$+ \int_{x_0}^{x_1} 2y^{\text{iv}} \delta y \, dx + \int_{x_0}^{x_1} \delta y''^2 \, dx. \tag{1}$$

We have also, as equations of transformation,

$$\left.\begin{aligned} y' = \frac{1}{x'}, \quad y'' = -\frac{x''}{x'^3}, \quad y''' = -\frac{x'''}{x'^4} + \frac{3x''^2}{x'^5}, \\ y^{\text{iv}} = -\frac{x^{\text{iv}}}{x'^5} + \frac{10x''x'''}{x'^6} - \frac{15x''^3}{x'^7}. \end{aligned}\right\} \tag{2}$$

Hence we shall have

$$U_{\prime} = \int_{y_0}^{y_1} \frac{x''^2}{x'^5} \, dy = \int_{y_0}^{y_1} V_{\prime} \, dy.$$

Varying $U_{\prime}$, the terms for the upper limit become

$$\left\{ -\frac{5x''^2}{x'^6} - \frac{d}{dy} \frac{2x''}{x'^5} \right\}_1 \delta x_1 + \left\{ \frac{2x''}{x'^5} \right\}_1 \delta x_1' =$$

$$\left\{ -\frac{2x'''}{x'^5} + \frac{5x''^2}{x'^6} \right\}_1 \delta x_1 + \left\{ \frac{2x''}{x'^5} \right\}_1 \delta x_1'. \tag{3}$$

Whence, after subtracting, as before, $V_1 \delta x_1$ or $\left\{ \frac{x''^2}{x'^5} \right\}_1 \delta x_1$, we have

$$\left\{ -\frac{2x'''}{x'^5} + \frac{4x''^2}{x'^6} \right\}_1 \delta x_1 + \left\{ \frac{2x''}{x'^5} \right\}_1 \delta x_1'. \tag{4}$$

But by equation (10) of the last article these terms should become

$$(2y'y''' - 2y''^2)_1 \delta x_1 - 2y_1'^2 y_1'' \delta x_1'; \tag{5}$$

and if by the aid of (2) we express (5) in terms of (4), it will become identical with (4). In like manner we might treat the terms at the lower limit, only adding $V_0 \delta x_0$. We also have

$$M = 2y^{\text{iv}}, \qquad N = 5 \frac{d}{dy} \frac{x''^2}{x'^6} + 2 \frac{d^2}{dy^2} \frac{x''}{x'^6}$$

$$= \frac{2x^{\text{iv}}}{x'^5} - \frac{20x''x'''}{x'^6} + \frac{30x''^3}{x'^7} = -2y^{\text{iv}},$$

as will appear by consulting the value of $y^{\text{iv}}$ given in (2), so that here also $N = -M$.

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## SECTION IX.

### *DISCONTINUOUS SOLUTIONS.*

**192.** We now enter upon a portion of our subject which is of comparatively recent development, but is nevertheless of the highest analytical importance. But some general view of the nature of discontinuous solutions having been presented in Art. 103, we shall, without further explanation, proceed at once to the consideration of an example to which the development of the subject is chiefly due.

### Problem XXXII.

*It is required to determine the form of the surface of revolution which shall meet the axis of revolution at two fixed points, have a given area, and enclose a maximum solid; the two fixed points being so taken as to render a sphere inadmissible.*

It will readily appear that this is merely Prob. XVI. with an additional restriction upon the limits, which can in no way

affect the general value of $U$. Hence, as there, we shall, assuming $x$ as the axis of revolution, have

$$U = \int_{x_0}^{x_1} \left(y^2 + 2ay\sqrt{1+y'^2}\right)dx = \int_{x_0}^{x_1} V dx;$$

and the limits being fixed, we must have, to the first order,

$$\delta U =$$

$$\int_{x_0}^{x_1} \left\{ 2y + 2a\sqrt{1+y'^2} - \frac{d}{dx}\frac{2ayy'}{\sqrt{1+y'^2}} \right\} \delta y\, dx = \int_{x_0}^{x_1} M\,\delta y\, dx. \quad (1)$$

Now if, as usual, we make $\delta U$ vanish, we must have

$$M = 2y + 2a\sqrt{1+y'^2} - \frac{d}{dx}\frac{2ayy'}{\sqrt{1+y'^2}} = 0. \quad (2)$$

To integrate this equation, write

$$2y\,dy + 2a\sqrt{1+y'^2}\,dy - 2ay'.\,d\frac{yy'}{\sqrt{1+y'^2}} = 0.$$

Now

$$\int 2a\sqrt{1+y'^2}\,dy = 2a\sqrt{1+y'^2}\,y - \int \frac{2ayy'}{\sqrt{1+y'^2}}\,dy'$$

and

$$-\int \frac{2ayy'}{\sqrt{1+y'^2}}\,dy' = -\frac{2ayy'}{\sqrt{1+y'^2}}y' + \int 2ay'.\,d\frac{yy'}{\sqrt{1+y'^2}}.$$

Hence, by reduction, (2) gives

$$y^2 + \frac{2ay}{\sqrt{1+y'^2}} = c; \quad (3)$$

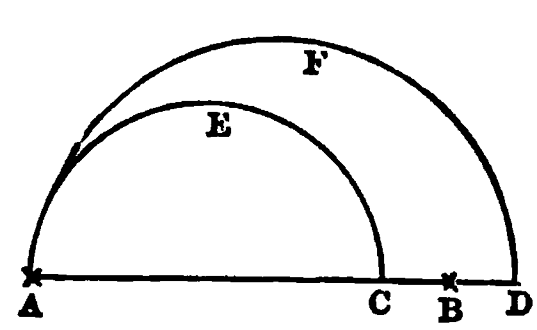
and since the curve must meet the axis of $x$, $c$ vanishes, and we must have

$$y^2 + \frac{2ay}{\sqrt{1+y'^2}} = y\left\{y + \frac{2a}{\sqrt{1+y'^2}}\right\} = 0. \qquad (4)$$

This is equation (4), Prob. XVI.; and if we make $\delta U$ zero, we are necessarily led to this equation.

**193.** But the equation at which we have now arrived admits of two solutions, $y = 0$ and $y + \frac{2a}{\sqrt{1+y'^2}} = 0$. The first, however, cannot hold throughout the entire range of integration, since the surface generated is to be a given finite area, while the second will, as we have already seen, lead to a sphere, having its centre on the axis of $x$, and is therefore excluded by the conditions of the problem.

We are naturally led next to inquire whether the solution sought might not be obtained by combining in some manner the preceding solutions. Thus, in the figure, let $A$ and $B$ be the two fixed points. Then if the given surface be less than $\pi\overline{AB}^2$, we may suppose the generating curve to be $AECB$; and when the surface exceeds $\pi\overline{AB}^2$, we may suppose the generatrix to be $AFDB$.




Under this supposition we know that the revolution of the semicircles will render the given integral, when taken from $A$ to $C$, or from $A$ to $D$, a maximum, while the line $CB$ or $DB$ may be considered as generating a cylinder whose diameter is infinitesimal, and whose surface and volume are so likewise.

Here, however, a new difficulty presents itself. For if in $M$

we substitute zero for $y$, observing that $y'$ will be zero also, we shall obtain $M = 2a$; so that it appears that $y = 0$ is not a solution of the equation $M = 0$, and that, therefore, if this latter equation is to hold throughout the entire range of integration, this solution must be abandoned also. The fact is, however, that we cannot reject the solution $y = 0$ because it does not satisfy the equation $M = 0$, since the question now involves a principle of variations which we have not hitherto considered; and this we next proceed to explain.

**194.** In former problems we have been obliged to consider $\delta y$ as capable of having either sign, and therefore, when $\delta U$ was developed into a series, and the terms of the first order transformed in the usual manner, we were compelled to equate $M$, and also the coefficients of $\delta y_1$, $\delta y_0$, $\delta y_1'$, etc., severally to zero, as the only means of preventing the terms of the first order from exceeding the sum of all the others, and thus rendering the sign of $\delta U$ positive or negative at pleasure. But by referring to Art. 99 we see that the conditions of this problem prevent $y$ from becoming negative, and hence when $y$ is zero—that is, when the primitive curve coincides with the axis of $x$—we can give $y$ positive increments only.

To determine in the most general manner what effect this restriction would produce when applied to the present problem, let us suppose that $U$ and $V$ retain the same form as before, but that the limiting values of $x$ and $y$ become variable, so that we shall have

$$[\delta U] = V_1 dx_1 - V_0 dx_0 + \left(\frac{2ayy'}{\sqrt{1+y'^2}}\right)_1 \delta y_1 - \left(\frac{2ayy'}{\sqrt{1+y'^2}}\right)_0 \delta y_0$$
$$+ \int_{x_0}^{x_1} M \delta y \, dx + \text{etc.}, \qquad (5)$$

where brackets denote the entire variation of $U$, the etc. the terms of an order higher than the first, and $M$ has the form

given in (2). If now we suppose $y$ to become zero throughout the range of integration, (5) will become

$$[\delta U] = \int_{x_0}^{x_1} 2a\,\delta y\,dx + \text{etc.} = \int_{x_0}^{x_1} - 2r\,\delta y\,dx + \text{etc.}, \quad (6)$$

where $r$ is a positive constant, since it appears, by referring to Art. 99, that $2a$ must be essentially negative.

Since the proposed solution $y = 0$ does not reduce the terms of the first order in $[\delta U]$ to zero, but merely to a single term, it is plain that this term will exceed the sum of all the following terms, and hence that its sign will control that of $[\delta U]$. But because $\delta y$ is now necessarily positive, the sign of this controlling term is no longer in our power, but is essentially negative, thus rendering $[\delta U]$ a negative quantity of the first order. Hence, if we wish to render $U$ a maximum or a minimum, not as compared with all consecutive states of $U$ which can be produced by varying $y$ and $y'$, but with such only as can be obtained by making $\delta y$ invariably positive or negative, we see that the solution $y = 0$ will, in the former case, render $U$ a maximum, but in the latter a minimum. We may call such maxima and minima *conditional maxima or minima.*

Now as the sign of $\delta U$ will depend upon that of the term of the first order, we have in this case nothing to do with the terms of the second order, and thus the problem is much simplified, unless, indeed, these terms should happen to become infinite, which would, as before, throw doubt upon the whole solution. But this will not occur in the present case.

We see, then, that in this case, by restricting $\delta y$ to one sign, we render it unnecessary that the proposed solution should reduce $M$ to zero, and also remove the necessity of an examination of the terms of the second order. Neither was it necessary that $M$ should become a constant, but merely that it should be finite and of invariable sign. But should $M$ change its sign, we could, in the same manner as has already been ex-

plained for terms of the second order, cause the term of the first order, and consequently $[\delta U]$, to assume either sign at our pleasure.

Simple as is the foregoing principle of restricting $\delta y$ to one sign, it appears to have been first introduced into the calculus of variations by Prof. Todhunter, in the *Philosophical Magazine* for June, 1866. It will, however, when somewhat more extended, afford the basis for some important investigations, and will also serve to explain some points which have hitherto been the source of difficulty to the student in this department of analysis.

**195.** In applying this principle to the present problem, let us first suppose the given surface to be less than that of a sphere having $AB$ as its diameter, and let the abscissa of $A$ be $x_0$, that of $B$, $x_1$, and that of $C$, $x_2$. Then suppose the integral to be divided into two parts, the first extending from $A$ to $C$, and the second from $C$ to $B$; so that we may write

$$U = \int_{x_0}^{x_2} V dx + \int_{x_2}^{x_1} V dx. \tag{7}$$

Now, supposing $M$ to be zero throughout the first integral, its variation will reduce to

$$\left(\frac{2ayy'}{\sqrt{1+y'^2}}\right)_2 \delta y_2;$$

and putting $y = 0$, $y' = \infty$, observing that $2a = -2r = -R$, $R$ being the radius of the sphere, this term will also vanish. If we vary this portion of the integral only, while leaving the rectilinear portion unvaried, we shall, theoretically, be obliged to examine the sign of the terms of the second order; and we have already seen that this investigation is not altogether satisfactory. Still, as it is well known apart from the calculus of variations that the sphere is the solid of maximum volume for

a given surface, we may assume that $U$ will in this case become a maximum; that is, that $[\delta U]$ will become a small negative quantity of the second order.

Now throughout the second integral we have

$$M = 2a = -2r = -R,$$

and the variation of this integral becomes $\int_{x_2}^{x_1} - R\,\delta y\,dx$, which must be negative, since $\delta y$ is invariably positive; and thus in this case the whole integral $U$ must become a maximum.

It should be observed that while the values of $x_2$ and $y_2$ are the same for both parts of the solution, those of $y'$ for the same point differ. Thus for the circle $y_2'$ is infinite, while for the rectilinear part $y_2'$ is zero, and we shall be obliged sometimes to observe this and similar distinctions with great care.

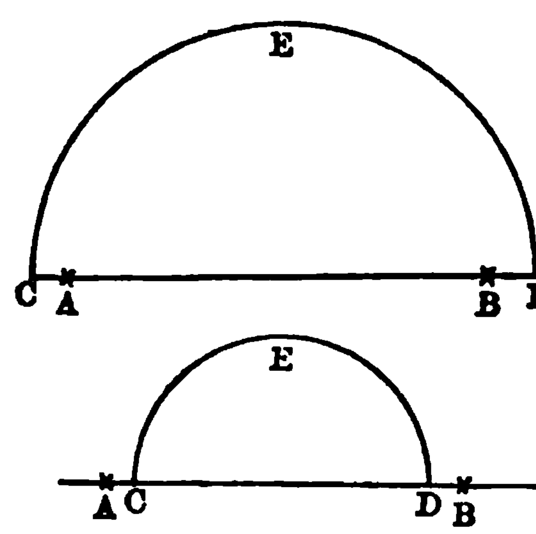
When the given surface exceeds that of a sphere described upon $AB$ as a diameter, let $x_2$ be the abscissa of $D$. Then we may consider $U$ as consisting of two integrals, the first extending from $A$ to $D$, and the second from $D$ to $B$, and we may still write

$$U = \int_{x_0}^{x_2} V dx + \int_{x_2}^{x_1} V dx. \tag{8}$$

This mode of considering the integral may seem erroneous, inasmuch as it will compel us to regard $x$ as doubling upon itself at $D$, and $x_1$, therefore, as greater than $x_2$. But it is to be observed that we assume that $U$ and $\delta U$ are continuous integrals—that is, that they are capable of being expressed by one definite integral—and this requires that $x$ shall be uninterrupted. Adopting for the present this view of the subject, we see, as before, that if we vary the arc $AD$ only, $\delta U$ must become a negative quantity of the second order, and that if we vary the line $DB$ also, we shall have $\delta U = \int_{x_2}^{x_1} - R\,\delta y\,dx$,

which is negative, as before. In this case we in reality reckon twice the volume generated by $\delta y$ along $DB$, when we pass to the derived solid.

We may, however, construct the two solutions as in the subjoined figure, in which case we shall be obliged to consider $U$ as consisting of three integrals, and we have therefore adopted the other construction as being more easily explained.




**196.** We have already shown, in Art. 101, that when the discontinuous solution is necessary, that necessity arises from the fact that the conditions which require the surface to be given, and the two terminal points on the axis of $x$ to be also assigned, have been so fulfilled as to render them incompatible with the general solution. Now we shall find, as we proceed, that discontinuous solutions generally, if not always, arise from some incompatibility in the conditions of the problem, and that the conflicting conditions are imposed sometimes consciously, that is explicitly, and sometimes unconsciously, that is implicitly. The present problem would, of itself, afford an example of the former kind, but it is in reality only a completion of the discussion suggested by Prob. XVI., and there the discontinuity was of the latter kind, arising from conditions incidentally imposed, the effects of which were not foreseen.

**197.** We will next consider an example which will serve to extend our theoretical knowledge, and to prepare us for the discussion of more important questions.

## Problem XXXIII.

*Let* $U = \int_{x_0}^{x_1} (y''^2 - 2y)dx = \int_{x_0}^{x_1} V dx$, *and let it be required to maximize or minimize* $U$, *the limiting values of* $x$ *being fixed, those of* $y$ *being zero, and it being also required that a certain fixed point, whose co-ordinates are* $x_2$ *and* $y_2$, *shall not fall without the required curve.*

This is, in fact, merely a restricted form of Prob. V., and we have

$$M = 2y^{\text{iv}} - 2. \qquad (1)$$

Now if it be possible to draw between the fixed points on the axis of $x$ a curve satisfying the equation $M = 0$, and also enclosing the point $x_2, y_2$, there will be no difficulty, and we shall have a minimum as in Prob. V. If the point should happen to fall upon the curve, $\delta y_2$ could not be made negative, but this would not affect the problem, since the curve would render $U$ a minimum for all admissible variations of $y$ and $y''$.

Suppose, however, that no curve satisfying the equation $M = 0$ can be drawn so as to enclose or pass through the point $x_2, y_2$. Still, as the sign of $\delta y$ is wholly unrestricted, except at the points $x_0$ and $x_1$, and also possibly at the point $x_2, y_2$, if the curve pass through that point, we feel sure that $M$ must vanish throughout the entire integral $U$. We are therefore naturally led to inquire whether the solution might not be furnished by drawing from $x_0$ and $x_1$ severally an arc of a curve satisfying the equation $M = 0$, the two arcs meeting and not excluding the point $x_2, y_2$; and such a solution we now proceed to consider.

**198.** Let the arcs meet at the point $x_3, y_3$; $x_2$ and $x_3$ being less than $x_1$. Then $U$ may be written

$$U = \int_{x_0}^{x_3} V dx + \int_{x_3}^{x_1} V dx, \qquad (2)$$

where $V$ has the same form as before. But although the two arcs satisfy the same differential equation $M = 0$, still the constants which enter their equations cannot be identical; otherwise they would form one and the same curve, which is contrary to our present supposition. Hence by making $M$ zero in (1), and integrating, the general equation of the two arcs may be written

$$\left.\begin{aligned} y &= \frac{x^4}{24} + cx^3 + c_1 x^2 + c_2 x + c_3, \\ Y &= \frac{x^4}{24} + gx^3 + g_1 x^2 + g_2 x + g_3. \end{aligned}\right\} \qquad (3)$$

If now to the first order we take the variation of each integral in $U$ separately, and transform it in the usual manner, observing that $\delta y_1$ and $\delta y_0$ vanish, and that the parts which remain under the sign of integration must also vanish, because $M$ is zero throughout $U$, we shall obtain

$$\begin{aligned} \delta U = -\,2y_2'''\delta y_2 + 2Y_2'''\delta Y_2 + 2Y_1''\delta Y_1' - 2y_0''\delta y_0' \\ + 2y_2''\delta y_2' - 2Y_2''\delta Y_2'. \end{aligned} \qquad (4)$$

Since all the variations in this equation are of unrestricted sign, $\delta U$ must vanish; and also if (4) be expressed so as to involve only variations which are entirely independent, the coefficients of these variations must severally vanish. Now if we assume that $x_2$ does not vary, $\delta y_2$ and $\delta Y_2$ are the same quantity. Moreover, from (3), we have

$$y_2''' = x_2 + 6c, \qquad \text{and} \qquad Y_2''' = x_2 + 6g. \qquad (5)$$

Whence

$$-\,2y_2'''\,\delta y_2 + 2Y_2'''\,\delta Y_2 = 12(g - c)\,\delta y_2; \qquad (6)$$

and since $\delta y_2$ is certainly an independent variation, $c$ and $g$ must be equal,

Now consider the terms $2y_3''\delta y_3' - 2Y_3''\delta Y_3'$. To make these terms vanish we may have $y_3'' = Y_3''$, and also $\delta y_3' = \delta Y_3'$, or $y_3'' = 0$ and $Y_3'' = 0$. Under the first supposition $y_3'$ and $Y_3'$ must mean the same thing, otherwise their variations would not be necessarily equal.

If now we equate $y_3''$ to $Y_3''$, $y_3'$ to $Y_3'$, and $y_3$ to $Y_3$, taking the values of these quantities found by differentiating (3), we easily discover that if $c$ and $g$ are equal, $c_1 = g_1$, $c_2 = g_2$ and $c_3 = g_3$, which is, as has been shown, not admissible in this case.

But suppose $y_3'' = 0$ and $Y_3'' = 0$. Then $y_3'$ and $Y_3'$ need not mean the same thing, and we have only $c = g$ and $c_1 = g_1$. But there still remain in (4) the terms $2Y_1''\delta Y_1' - 2y_0''\delta y_0'$, and to make these vanish we must have $Y_1'' = 0$ and $y_0'' = 0$. Take the origin midway between the points $x_0$ and $x_1$, and let $x_1 = e$ and $x_0 = -e$. Equating the values of $y_0''$ and $Y_1''$, as found from (3), we have

$$\frac{e^2}{2} + 6ge + 2g_1 = \frac{e^2}{2} - 6ce + 2c_1.$$

Whence, since $c = g$ and $c_1 = g_1$, $c$ and $g$ are zero.

The equation $Y_1'' = 0$ now becomes $\dfrac{e^2}{2} + 2g_1 = 0$, and the equation $Y_3'' = 0$ gives $\dfrac{x_3^2}{2} + 2g_1 = 0$, impossible equations unless $e^2$ and $x_3^2$ be equal, which they cannot be since $x_3$ was taken numerically less than $x_1$. Hence we must abandon this solution, since it will neither cause $\delta U$ to the first order to vanish nor to have an invariable sign.

**199.** There remains but one supposition, which is that the arcs be drawn as before, but meet at the point $x_2$, $y_2$; and this, which we shall find to be the correct solution, we next proceed to consider.

It is plain that $U$ and also $\delta U$ will have the same form as before, except that the suffix 3 will be changed into 2. But it

will not now be necessary to make all the terms in $\delta U$ vanish, because $\delta y_2$ and $\delta Y_2$ are the same quantity, and $\delta y_2$ is necessarily positive. Now, as before, the terms involving these two variations will become $12(g-c)\,\delta y_2$, and it will be necessary to make the remainder of $\delta U$ vanish, because it contains only variations of unrestricted sign.

We have then to examine whether we can make these terms vanish; and if so, what will then be the sign of $g-c$.

To make the terms involving $\delta y_2'$ and $\delta Y_2'$ vanish, we have, as before,

$$Y_2''=y_2'' \qquad \text{and} \qquad Y_2'=y_2'. \tag{7}$$

The first equation being necessary, and the second also necessary unless $Y_2''=0$ and $y_2''=0$, a case which we shall subsequently consider.

To make the terms involving $\delta Y_1'$ and $\delta y_0'$ vanish, we have, as before,

$$Y_1''=0 \qquad \text{and} \qquad y_0''=0. \tag{8}$$

We have also, from the other conditions of the question,

$$y_0=0, \qquad Y_1=0, \qquad Y_2=y_2; \tag{9}$$

and these equations, together with (3), which still holds, will be sufficient for our purpose.

**200.** Take the origin as before, and also denote $x_2$ by $a$, and $y_2$ by $b$. Then finding, from (3), the successive differentials of $y$ and $Y$, (7), (8), and (9) give, by substitution,

$$\left.\begin{aligned} \frac{a^2}{2}+6ac+2c_1&=\frac{a^2}{2}+6ag+2g_1, \\ \frac{a^3}{6}+3a^2c+2ac_1+c_2&=\frac{a^3}{6}+3a^2g+2ag_1+g_2; \end{aligned}\right\} \tag{10}$$

$$\frac{e^2}{2} + 6ge + 2g_1 = 0, \qquad \frac{e^2}{2} - 6ce + 2c_1 = 0; \qquad (11)$$

$$\left.\begin{aligned} &\frac{e^4}{24} + ge^3 + g_1e^2 + g_2e + g_3 = 0, \\ &\frac{e^4}{24} - ce^3 + c_1e^2 - c_2e + c_3 = 0; \end{aligned}\right\} \qquad (12)$$

$$\left.\begin{aligned} &\frac{a^4}{24} + ga^3 + g_1a^2 + g_2a + g_3 = b, \\ &\frac{a^4}{24} + ca^3 + c_1a^2 + c_2a + c_3 = b. \end{aligned}\right\} \qquad (13)$$

Now from (11) we have

$$g_1 = -3ge - \frac{e^2}{4}, \qquad \text{and} \qquad c_1 = 3ce - \frac{e^2}{4}, \qquad (14)$$

and the first of equations (10) becomes

$$g(a - e) = c(a + e); \qquad (15)$$

and solving for $ge$, and then adding $ge$ to both members, we can obtain

$$g - c = \frac{2eg}{a + e}. \qquad (16)$$

But $e$ is positive; and estimating $x$ toward $a$, $a$ will also be positive, so that the sign of $g - c$ will be the same as that of $g$, which we must next determine.

Subtracting (12) from (13) and dividing by $a - e$, we have

$$g_2 + g_1(a + e) + g(a^2 + ae + e^2) + \frac{a^3 + a^2e + ae^2 + e^3}{24} = \frac{b}{a - e}$$

$$= g_2 + g(a^2 - 2ae - 2e^2) + \frac{a^3 + a^2e - 5ae^2 - 5e^3}{24}, \qquad (17)$$

the last member being obtained by substituting for $g_1$ its value from (14). In like manner, from (12) and (13), we obtain

$$c_2 + c(a^2 + 2ae - 2e^2) + \frac{a^3 - a^2e - 5ae^2 + 5e^3}{24} = \frac{b}{a+e}. \quad (18)$$

Now subtracting (18) from (17), substituting for $c$ its value $\frac{g(a-e)}{a+e}$ from (15), and reducing the second member to a common denominator, we have

$$g_2 - c_2 + g\left\{a^2 + 2ae - 2e^2 - \frac{a-e}{a+e}(a^2 + 2ae - 2e^2)\right\}$$

$$+ \frac{a^2e - 5e^3}{12} = \frac{2be}{a^2 - e^2}$$

$$= g_2 - c_2 - g\frac{2a^2e + 4e^3}{a+e} + \frac{a^2e - 5e^3}{12}. \quad (19)$$

From (14) and (15) we have

$$c_1 - g_1 = 3e(c+g), \quad (20)$$

$$c + g = \frac{2ag}{a+e}. \quad (21)$$

Now from the second of equations (10), we obtain

$$g_2 - c_2 = 2a(c_1 - g_1) + 3a^2(c - g)$$

$$= 6ae(g+c) - 3a^2(g-c) = \frac{6a^2eg}{a+e}, \quad (22)$$

the second member being obtained by (20), and the third by (21) and (16). Hence, by substitution, (19) becomes

$$4g\frac{a^2e - e^3}{a+e} + \frac{a^2e - 5e^3}{12} = \frac{2be}{a^2 - e^2}.$$

Whence

$$4g(e-a)=\frac{2b}{e^2-a^2}-\frac{5e^2-a^2}{12}.$$

Now we will so estimate $y$ as to make $b$ positive. Then, since $e$ exceeds $a$, to make $g$ positive, we must have

$$2b>(5e^2-a^2)\frac{e^2-a^2}{12},$$

or

$$b>\frac{a^4}{24}-\frac{a^2e^2}{4}+\frac{5e^4}{24}. \tag{23}$$

But in Art. 42 we showed that the general equation of the single curve meeting $x$ at the points $-e$ and $+e$, $y_1'$ and $y_0'$ not being fixed, is

$$y=\frac{x^4}{24}-\frac{e^2x^2}{4}+\frac{5e^4}{24}.$$

Hence, if the members of (23) were equal, $b$, or $y_2$, would be an ordinate of a single curve drawn from the point $x_0$ to $x_1$, and satisfying all the other conditions of the problem. But since $y_2$ or $b$ is in this case too great to be made the ordinate of any such single curve, the conditions of (23) are fulfilled, and $g$ must be positive.

Therefore, in this case, $\delta U$ reduces to $12(g-c)\delta y_2$, which, because $\delta y_2$ cannot become negative, is positive, so that these arcs furnish the minimum solution required.

**201.** We have still to consider the case in which $y_2''=0$ and $Y_2''=0$. Here the second of equations (10) is not necessarily true, but the first may be replaced by the two equations

$$\frac{a^2}{2}+6ag+2g_1=0 \quad \text{and} \quad \frac{a^2}{2}+6ac+2c_1=0. \tag{24}$$

Then from (11) and (24) we readily deduce

$$g = -\frac{a+e}{12} \quad \text{and} \quad c = -\frac{a-e}{12}.$$

Hence $\delta U$ becomes in this case

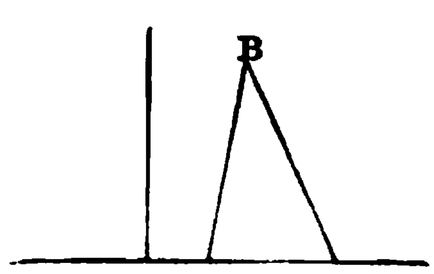
$$\delta U = 12(g - c)\,\delta y_2 = -2e\,\delta y_2,$$

which, as $\delta y_2$ is still positive, must be negative.

Thus it appears that this solution will render $U$ a maximum. Now in the present case $g$ is necessarily negative, and we have seen that when $Y_2'' = y_2''$ and $Y_2' = y_2'$, $g$ is necessarily positive. Hence, when we satisfy the first of these equations by making $Y_2''$ and $y_2''$ severally vanish, the second cannot hold true. We see, then, that the minimum solution consists of two arcs which satisfy the equation $M = 0$ and meet at the point $x_2, y_2$, so as to have there no abrupt change of direction, and to make their radii of curvature at that point equal and finite.

**202.** In closing this discussion we must observe, first, that when we propose to make the two arcs meet at the point $x_2, y_2$, it is by no means the same as if we had been required to draw each arc so as always to pass through the two fixed points. For then $y_2$ would have no variation, and we would treat each curve separately by the well-known rules of variations.

In the second place, the terms maxima and minima are here also used in the technical sense already explained, and we must be careful not to say that the present solution gives the least value of $U$. For $B$ being the point $x_2$, $y_2$, we can, by a construction like that of the figure, make $U$ as small as we please.



All that follows from the preceding discussion is, that if we draw two arcs as required by the solution, and then, regarding this curved line as a primitive, pass to any other curved line which can be derived from the first by infinitesimal variations of $y$ and $y''$, the variation of $y_2$ being positive, $U$ will be thereby increased by a quantity of the first order. If we make $\delta y_2$ zero, the proposed solution will reduce the terms of the first order to zero, and we shall be obliged as usual to appeal to those of the second order, which will be

$$\delta U = \int_{x_0}^{x_2} \delta y''^2 \, dx + \int_{x_2}^{x_1} \delta Y''^2 \, dx,$$

which, being positive, will render $U$ in this case also a minimum.

**203.** We may now with profit consider partially the general theory of discontinuous solutions.

Suppose we wish to determine the relations between $x$ and $y$ which will maximize or minimize the expression $U = \int_{x_0}^{x_1} V dx$, $V$ being, as usual, any function of $x, y, y'$, etc. Then, after the usual transformation, we may write

$$\delta U = L_1 - L_0 + \int_{x_0}^{x_1} M \, \delta y \, dx,$$

where $L_1$ and $L_0$ have the well-known form of the terms at the limits. Now if no restriction be imposed upon $\delta y$, we know that $M$ must vanish throughout the entire range of integration, and likewise $L_1$ and $L_0$ must vanish.

But suppose the problem be such that $\delta y$ must always be positive or always negative; then it may not be necessary to make $M$ vanish, provided it be of invariable sign, and provided, also, that the terms at the limits either vanish or become of the same sign as the unintegrated part; in which case

$\delta U$ will become a quantity of the first order, and there will be no need of examining the terms of the second order.

Suppose, next, that $U$ is such that it may naturally be divided into a number of integrals, say $n$, the first extending from $x_0$ to $x_2$, the second from $x_2$ to $x_4$, etc., the last extending from $x_{2n-1}$ to $x_1$; and suppose $\delta y$ is of invariable sign throughout one or more of the intervals into which $x$ is divided, but is unrestricted throughout the others. Then $M$ must vanish throughout the latter; but if throughout each of the former $M$ be of invariable sign, and if the sign of $M\delta y$ be the same throughout each, $M$ need not vanish provided certain conditions can be secured at the limits, and we shall have a discontinuous solution, made up of curves satisfying different differential equations.

But when the sign of $\delta y$ becomes necessarily invariable throughout any interval, we shall find that this restriction results from the fact that there is throughout that interval some boundary which the required curve is forbidden to pass; and in order that the sign of $\delta y$ may be made invariable by this boundary, the required curve must, throughout that interval, coincide with it. It will, therefore, readily appear that whenever any portion of the required solution does not satisfy the equation $M = 0$, it can consist of nothing but a portion of some boundary, the nature of which will be generally known. Thus in the case of a sphere, this boundary, although not explicitly assigned, is easily seen to be the axis of $x$, the implicit condition that $y$ is not to become negative making this the boundary below which $y$ cannot pass.

If, however, the sign of $\delta y$ be restricted for some point or points only, as in the preceding problem, the equation $M = 0$ must hold throughout $U$, although the equations of the arcs for different intervals may differ widely in the values of the constants which they contain.

It will, we think, now be evident that, in general, when a discontinuous solution presents itself, it will be made up in

one of these three ways: first, some combination of arcs satisfying the equation $M = 0$; second, some boundary or certain boundaries; third, some combination of this boundary or these boundaries, with arcs satisfying the equation $M = 0$.

**204.** Let us now consider the integrated part of $\delta U$, when $U$ is divided as explained above.

As the different portions of the discontinuous solution meet at the points whose abscissæ are $x_2$, $x_4$, etc., $y$ will have the same value for two curves meeting at those points, but the values of $y'$, $y''$, etc., for two curves at their points of meeting may differ widely. To recognize this distinction, we employ the suffixes 2 and 3 to denote quantities both of which correspond to $x_2$, but belong to different curves meeting at the point $x_2, y_2$, and we divide $x$ into $x_0$, $x_2$, $x_4$, etc., the last being $x_1$, the suffixes 3, 5, etc., being reserved for the second of the two quantities corresponding to $x_2$, $x_4$, etc.

Now performing the integration for each integral separately, the first gives $L_2 - L_1$, the second $L_4 - L_3$, etc., so that the entire integrated part of $\delta U$ becomes

$$L_1 - L_0 + L_2 - L_3 + L_4 - L_5 + \text{etc.} + L_{2n-2} - L_{2n-1}, \text{ or } L. \quad (1)$$

Now if all the variations involved in $L$ be of unrestricted sign, it must vanish; and also if $L$ be transformed so as to contain independent variations only, the coefficients of these variations must severally vanish. But suppose some of the variations involved in $L$ to be of restricted sign. Then, the other terms having vanished as before, it may not be necessary to make these terms vanish also. For if these restricted variations be related, suppose them to have been reduced to independent variations, and let $H$, $I$, $K$, etc., be the several products of each variation and its coefficient. Then we can reduce any of the quantities $H$, $I$, etc., to zero by making its variation factor vanish. If, therefore, these quantities be all of like sign, that of $L$ is determined; but if, on the contrary, they be not, $L$ can be made positive or negative according as

we reduce to zero the negative or positive quantities. But suppose $H$, $I$, etc., to be of like sign, making that of $L$ the same, and that this sign does not conflict with that of $M\delta y$ in the unintegrated part. Then $\delta U$ becomes a quantity of the first order, having a fixed sign, and we need not examine the terms of the second order. But if $H$, $I$, etc., be of unlike sign, or if their sign, when the same, conflict with that of $M\delta y$, $L$ must vanish altogether.

In equation (1) we have assumed that no portion of the axis of $x$ is to be counted twice, as in Prob. XXXII., where the sphere extends beyond $x_1$, because such cases will seldom occur. When, however, they do arise, $L$, although differing somewhat in form from (1), can be readily found by integrating each portion separately, as before; and then all the conditions which we have just explained will hold true for this case also.

### Problem XXXIV.

**205.** *It is required to determine what will be the solution of Prob. VII. when the two fixed points are so taken that no catenary can be drawn between them having its directrix on the axis of $x$.*

Here

$$U = \int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx = \int_{x_0}^{x_1} V dx \tag{1}$$

and

$$M = \sqrt{1+y'^2} - \left(\frac{yy'}{\sqrt{1+y'^2}}\right)'. \tag{2}$$

Now it is natural to inquire, first, whether any restrictions have, either explicitly or implicitly, been imposed upon the sign of $\delta y$, in virtue of which the equation $M = 0$ need not hold throughout $U$. For if not, the solution can consist of nothing that will not satisfy this equation. Now $Vdx$ in (1) is the

value of any element of the generated surface divided by $2\pi$, and it does not seem reasonable to suppose that this surface can ever become negative. Hence, since we take $\sqrt{1+y'^2}$ positively, it would appear that $y$ cannot be negative; that is, that $\delta y$, along the axis of $x$, must be positive.

We infer, then, that should the solution contain anything which does not satisfy the equation $M = 0$, it can only be some portion of the axis of $x$, and that such portions will be likely to occur.

**206.** Let us next examine the equation $M = 0$ to see what can be obtained from this source.

This equation will give

$$\sqrt{1+y'^2}\,dy - y'd\frac{yy'}{\sqrt{1+y'^2}} = 0,$$

$$\int \sqrt{1+y'^2}\,dy = \sqrt{1+y'^2}\,y - \int \frac{yy'}{\sqrt{1+y'^2}}\,dy'$$

$$= y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} + \int y'd\frac{yy'}{\sqrt{1+y'^2}} + c.$$

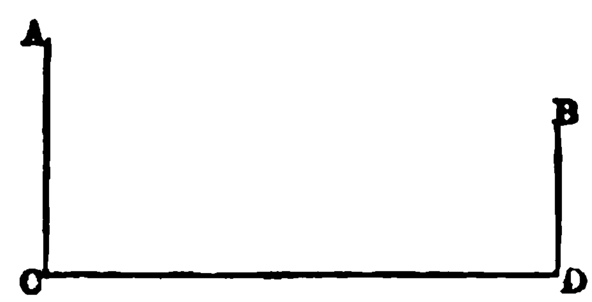
Whence

$$\frac{y}{\sqrt{1+y'^2}} = c. \tag{3}$$

This is the same as equation (2), Art. 59, so that this is the only condition which can be obtained from the equation $M = 0$.

Suppose now that we digress from the method of solution pursued in Art. 59, and make $c$ zero. Then (3) will give either $y = 0$ or $y' = \infty$, and these two solutions, although neither can be employed alone, can be combined. For let $A$ and $B$ of the figure be the two fixed points, $CD$ being the axis of $x$. Then the discontinuous solution proposed will be the broken line $ACDB$.

Thus in this case, as in Prob. XXXII., the solution $y = 0$, which arises from the same conditions in both, is suggested as one solution of the equation $M = 0$, which it does not, however, in either case satisfy. But this suggestion was not




necessary, as this solution was anticipated by the reasoning of the preceding article, which would be equally applicable to Prob. XXXII.

**207.** We now proceed to show that the proposed solution will minimize $U$. As we cannot treat infinite quantities by the methods of variations, we shall, to avoid their occurrence, transform to polar co-ordinates. Take some point within the figure as the pole, the initial line being parallel to $CD$. Let $v$, the angle between $r$, the radius vector, and this initial, be estimated in the direction $ACDB$, and let $k$ be the distance of the pole from $CD$. Then any element of the generating curve will be $\sqrt{r^2 + r'^2}\,dv$, and its distance from $CD$ will be $k - r \sin v$. Then, $ds$ being an element of the surface,

$$ds = 2\pi(k - r\sin v)\sqrt{r^2 + r'^2}\,dv.$$

Whence

$$U = \int_{v_0}^{v_1} (k - r\sin v)\sqrt{r^2 + r'^2}\,dv = \int_{v_0}^{v_1} V dv \tag{4}$$

and

$$\delta U =$$

$$\int_{v_0}^{v_1} \left\{ -\sqrt{r^2 + r'^2}\sin v\,\delta r + \frac{k - r\sin v}{\sqrt{r^2 + r'^2}} r\,\delta r + \frac{k - r\sin v}{\sqrt{r^2 + r'^2}} r'\delta r' \right\} dv$$

$$= \int_{v_0}^{v_1} \left\{ -\sqrt{r^2 + r'^2}\sin v\,\delta r + zr\,\delta r + zr'\delta r' \right\} dv. \tag{5}$$

But since the proposed solution cannot be represented by the same equation throughout, (3) and (4) must, without in any manner changing their form, be written as three integrals—that is, three times with different limits—the first portion, $AC$, extending from $v_0$ to $v_2$; the second, $CD$, from $v_3$ to $v_4$; and the third, $DB$, from $v_4$ to $v_1$. Then, transforming $\delta U$ in the usual manner, we have

$$\delta U = (zr'\delta r)_1 - (zr'\delta r)_0 + (zr'\delta r)_2 - (zr'\delta r)_3 + (zr'\delta r)_4 - (zr'\delta r)_5$$

$$+ \int_{v_0}^{v_1} \left\{ - \sqrt{r^2 + r'^2} \sin v + zr - \frac{d}{dx} zr' \right\} \delta r \, dv. \qquad (6)$$

Now the suffixes 2 and 3 relate to $C$ as being on the two lines $AC$ and $CD$, and the same is true of the suffixes 4 and 5, so that $r_2 = r_3$, $v_2 = v_3$, $r_4 = r_5$, $v_4 = v_5$, $\delta r_2 = \delta r_3$, $\delta r_4 = \delta r_5$; while $r_2'$ and $r_3'$ differ, as do also $r_4'$ and $r_5'$. Now $\delta r_1$ and $\delta r_0$ are zero, the points $A$ and $B$ being fixed; and although the other suffixed variations need not vanish, still at the points $C$ and $D$, for either line, we have $k - r \sin v = 0$; so that all the integrated terms in (6) disappear, and we have left only the integral, which must be considered as divided into three parts, as just explained.

**208.** We may now write

$$\delta U = \int_{v_0}^{v_2} M \delta r \, dv + \int_{v_3}^{v_4} M \delta r \, dv + \int_{v_4}^{v_1} M \delta r \, dv, \qquad (7)$$

where

$$M = - \sqrt{r^2 + r'^2} \sin v + zr - \frac{d}{dv} zr'. \qquad (8)$$

Now consider first the second integral. Along $CD$ we have $k - r \sin v = 0$, so that $M = - \sqrt{r^2 + r'^2} \sin v$, a negative quantity of invariable sign. But along this line $\delta r$ is always

negative, so that every element of this integral becomes a small positive quantity of the first order.

Let us now examine the sign of the first integral. Along $AC$ we have $r \cos v$ a constant, say $c$, so that we find

$$r' = \frac{r \sin v}{\cos v} = \frac{c \sin v}{\cos^2 v}, \qquad \sqrt{r^2 + r'^2} = \frac{c}{\cos^2 v},$$

$$zr = \frac{kr \cos^2 v}{c} - \frac{r^2 \sin v \cos^2 v}{c} = \frac{kr \cos^2 v - r^2 \sin v \cos^2 v}{r \cos v}$$

$$= k \cos v - r \cos v \sin v = k \cos v - c \sin v.$$

Whence

$$M = -\frac{c \sin v}{\cos^2 v} + k \cos v - c \sin v$$

$$- \frac{d}{dv} \left\{ (k \cos v - c \sin v) \frac{\sin v}{\cos v} \right\}$$

$$= -\frac{c \sin v}{\cos^2 v} - c \sin v + k \cos v - \frac{d}{dv} \left\{ k \sin v - \frac{c \sin^2 v}{\cos v} \right\}.$$

Differentiating the first term within the parenthesis, and also putting for $\sin^2 v$, $1 - \cos^2 v$, we may write

$$M = -\frac{c \sin v}{\cos^2 v} - c \sin v + \frac{d}{dv} \left\{ \frac{c}{\cos v} - c \cos v \right\},$$

which will be found by differentiation to reduce to zero. Similarly we shall find that $M$ will vanish along the line $BD$; so that if we vary the whole line $ACDB$, or $CD$ only, $\delta U$ will become a positive quantity of the first order, and we have a minimum.

**209.** If, however, the line $CD$ be not varied, we cannot, since the terms of the first order vanish along $AC$ and $BD$,

assert that $U$ will be a minimum without examining the terms of the second order; and this we next proceed to do.

Putting $u$ for $\sqrt{r^2+r'^2}$ and $Z$ for $k-r\sin v$, these terms are

$$\delta U=\frac{1}{2}\int_{v_0}^{v_1}\left\{\left[\frac{-2r\sin v}{u}+\frac{Zr'^2}{u^3}\right]\delta r^2\right.$$

$$\left.+\left[\frac{-2r'\sin v}{u}-\frac{2Zrr'}{u^3}\right]\delta r\,\delta r'+\frac{Zr^2}{u^3}\delta r'^2\right\}dv$$

$$=-\int_{v_0}^{v_1}\left\{\frac{r\sin v}{u}\delta r^2+\frac{r'\sin v}{u}\delta r\,\delta r'\right\}dv$$

$$+\frac{1}{2}\int_{v_0}^{v_1}\frac{Z(r'\delta r-r\delta r')^2}{u^3}\delta v. \tag{9}$$

Now the second integral in (9) can never be negative, so that we need only transform the first. We have

$$\int\frac{r'\sin v}{u}\delta r\,\delta r'dv=\frac{\delta r^2r'\sin v}{u}-\int\delta r\frac{d}{dv}\frac{r'\sin v\,\delta r}{u}.dv$$

$$=\frac{\delta r^2r'\sin v}{u}-\int\frac{r'\sin v}{u}\delta r\,\delta r'dv-\int\delta r^2\frac{d}{dv}\frac{r'\sin v}{u}.dv.$$

$$2\int\frac{r'\sin v}{u}\delta r\,\delta r'dv=\frac{\delta r^2r'\sin v}{u}-\int\delta r^2\frac{d}{dv}\frac{r'\sin v}{u}.dv. \tag{10}$$

Now we must observe that each integral is to be considered as divided into three integrals, identical in form but with different limits, so that the term $\frac{r'\sin v}{u}\delta r^2$ will, as usual, appear with the suffixes 0, 1, 2, 3, 4 and 5. But since $\delta r$ now becomes zero at the points $C$ and $D$, as well as at $A$ and $B$,

all these limiting terms must vanish, and then by the aid of (10) the first integral in (9) becomes

$$-\int_{v_0}^{v_1}\left\{\frac{r\sin v}{u}-\frac{1}{2}\frac{d}{dv}\frac{r'\sin v}{u}\right\}\delta r^2 dv=-\int_{v_0}^{v_1}N\,\delta r^2 dv. \quad (11)$$

Now along $AC$ we have, as before,

$$r\cos v=c, \qquad r=\frac{c}{\cos v}, \qquad r'=\frac{c\sin v}{\cos^2 v}, \qquad u=\frac{c}{\cos^2 v}, \quad (12)$$

so that along this line we have

$$N=\sin v\cos v-\frac{1}{2}\frac{d}{dv}\cos^2 v=0,$$

and similarly, $N$ will also vanish along $DB$, because there we shall have $r\cos v=-c$. Thus, finally, since $\delta r$ and $\delta r'$ are zero along $CD$, the terms of the second order reduce to the second integral in (9), which, as we have already seen, can never become negative.

**210.** It is plain that the discontinuous solution which we have just examined exists even when the fixed points are so taken that a catenary is admissible. But to determine in this case which of the minima gives to $U$ the smaller value is a problem of the differential and integral calculus solely, and for this purpose we have the following formulæ. For the discontinuous,

$$s=\pi(y_0^2+y_1^2),$$

$s$ being the entire surface. For the continuous, let $PT$ be any line tangent to the catenary at $P$, and meeting the axis of $x$ at $T$, the abscissa of which is $x_2$, and let $S$ denote the surface generated by $PT$, while $s$ denotes that generated by the portion of the catenary between $P$ and its lowest point. Then,

regarding $x_2$ as positive or negative according as $P$ and $T$ are on the same or opposite sides of the axis of $y$, when it passes through the lowest point, we have

$$s = S + \pi a x_2,$$

in which $a$ is the well-known constant of the equation of the catenary, and can be calculated approximately when the co-ordinates of the fixed points are given. It will be found, however, that sometimes the continuous, and sometimes the discontinuous solution will generate the smaller surface.

**211.** We have seen by the reasoning of Art. 5 that the calculus of variations is not theoretically bound to furnish all possible solutions; and since two may exist in the present problem, it is natural to inquire whether there may not be another, which will render the surface less than does either of those which we have considered. We reply that, while theoretically such might be the case, still no such solution has ever been discovered, and there would seem to be little doubt that one of the two already examined will always give the least, as well as a minimum value of $U$.

In fact, we are now beginning, and shall continue, to verify the remarks of Art. 14, and to show that, although subject to some restrictions which would seem to greatly limit its power, the calculus of variations is in reality capable of furnishing nearly all the solutions pertaining to the maxima and minima states of irreducible integrals. We shall find, moreover, that these solutions will generally in some way present themselves as solutions of the equation $M = 0$, although they may in reality not satisfy that equation at all.

### Problem XXXV.

**212.** *A projectile which is acted upon by gravity alone is to start from one fixed point and to pass through another. It is required to determine the nature of its path, so that the action may be a minimum.**

Assume the origin at the starting-point, and estimate $x$ vertically downward, and let the initial velocity be $\sqrt{2ga}$. Then we know that the velocity of the projected particle at any point of its path will be $\sqrt{2ga(x+a)}$. Hence

$$U = \int_{x_0}^{x_1} \sqrt{(x+a)(1+y'^2)}\, dx = \int_{x_0}^{x_1} V dx. \qquad (1)$$

Whence, in the usual way, we obtain

$$\sqrt{x+a}\,\frac{y'}{\sqrt{1+y'^2}} = \sqrt{c}. \qquad (2)$$

So that

$$y'^2 = \frac{c}{x+a-c},$$

and, by integration,

$$y - c_1 = \pm 2\sqrt{c(x+a-c)}. \qquad (3)$$

Since $x$ and $y$ are simultaneously zero at the starting-point, we have

$$-c_1 = \pm 2\sqrt{c(a-c)};$$

and (3) becomes

$$y \pm 2\sqrt{c(a-c)} = \pm 2\sqrt{c(x+a-c)}. \qquad (4)$$

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* See Todhunter's Researches, Chap. VIII.

If $c(a-c)$ be positive, (4) represents two parabolas, and we must now consider whether these parabolas can be made to pass through the second point. We have

$$y_1 \pm 2\sqrt{c(a-c)} = \pm 2\sqrt{c(x_1+a-c)}, \tag{5}$$

so that, squaring, we obtain

$$y_1^2 \pm 4y_1\sqrt{c(a-c)} = 4cx_1.$$

Whence

$$(y_1^2 - 4cx_1)^2 = 16y_1^2c(a-c).$$

Hence it appears that $c(a-c)$ can never become negative, and (5) will therefore contain no imaginary quantity, unless $c$ become imaginary. But the last equation may be written thus:

$$16c^2(x_1^2+y_1^2) - 8cy_1^2(x_1+2a) + y_1^4 = 0;$$

or, dividing by the coefficient of $c^2$, it may be written

$$c^2 - 2Pc + Q = 0. \tag{6}$$

From (6) we may obtain the values of $c$, which, since $P$ and $Q$ are positive, will be real so long as $Q$ does not exceed $P^2$, the two roots being equal when $P^2 = Q$. Now the condition $P^2 >$ or $= Q$ gives, by reduction,

$$y_1^2 < \text{ or } = 4a(x_1+a). \tag{7}$$

Hence we see that if the first member of (7) exceed the second, $c$ in (5) can have no real value; if the members become equal, $c$ can have but one value; and if the first member become less than the second, $c$ can have two real values.

Now it is evident that for any given values of $x_1$, $y_1$ and $a$, but one of the forms of (5) can be true for the same value of $c$, and that therefore we can have, passing through the two

fixed points, as many parabolas as there are real values for $c$. But (7), when its members are equal, is itself the equation of a parabola, which may be called the limiting parabola. For we see that if the second point lie without this parabola, it cannot be joined to the first by any parabola which will satisfy all the conditions of the question, so that the solution, if one exists, must be discontinuous. If the point be on this parabola, one parabola only can be drawn; while if the point be within the limiting parabola, two parabolas can be drawn.

Of course when the values of $x_1$, $y_1$ and $a$ are fully given, we can determine the one or two equations involved in (5), so that they may be without ambiguity of sign. But when two values of $c$ exist, we cannot determine which must be taken, unless we fix the angle which the projectile in starting makes with the horizontal, two angles being admissible.

**213.** Let us now examine the terms of the second order. We have

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \frac{\sqrt{x+a}}{\sqrt{(1+y'^2)^3}}\,\delta y'^2\,dx. \tag{8}$$

Now when $x$ decreases algebraically—that is, when the projectile is ascending—we must regard the velocity as negative. But then $ds$ is also negative, so that, both radicals in (8) becoming negative, $\delta U$ will be positive. When $x$ increases—that is, when the projectile is descending—both radicals become positive, so that $\delta U$ is positive.

If, then, the arc of the parabola with which we are concerned does not include the vertex, we undoubtedly have a minimum; but if we are required to reach or pass beyond the vertex, then, since $y'$ at that point becomes infinite, our conclusion that we shall have still a minimum cannot be regarded as altogether trustworthy, and we shall be obliged to resort to another method of investigation.

**214.** Let us now assume the horizontal as the axis of $x$, estimating $y$ vertically downward, and taking the origin at the vertical distance $a$ above the first fixed point. Then we shall have

$$U = \int_{x_0}^{x_1} \sqrt{y(1 + y'^2)}\, dx = \int_{x_0}^{x_1} V dx. \tag{9}$$

Whence, by formula (C), Art. 56, we have

$$\sqrt{y(1+y'^2)} = \frac{y' \sqrt{y}\, y'}{\sqrt{1+y'^2}} + \sqrt{C_1}.$$

Hence

$$\frac{y}{1+y'^2} = C_1, \qquad \frac{dx}{dy} = \pm \frac{C_1}{\sqrt{y - C_1}};$$

so that

$$x = \pm 2C_1 \sqrt{y - C_1} + C_2.$$

Whence

$$y = C_1 + \frac{(x - C_2)^2}{4C_1}. \tag{10}$$

Differentiating (10), we have

$$y' = \frac{x - C_2}{2C_1}. \tag{11}$$

Now (10) is the equation of a parabola when the directrix is taken as the axis of $x$, the origin being assumed at pleasure, and $C_2$ is the abscissa, while $C_1$ is the ordinate of the vertex, $4C_1$ being twice the parameter, or $2p$. For making $y'$ zero in (11), we have $x = C_2$, and then (10) gives, for the same point of the curve, $y = C_1$. Now, changing the origin to the point $C_2$, $C_1$, we shall obtain, after interchanging the variables $x$ and $y$, $y^2 = 4C_1x = 2px$. Hence we see that the distance of the directrix above the starting-point is always numerically equal to $a$, or the height due to the initial velocity.

Now we know that the focus of any parabolic path described by a projectile moving from $A$ to $B$ must be at the intersection of two circular arcs, described with the same radius $a$ from the two points respectively as centres. But if $a$ be so assumed that these circles cannot touch, there can be no continuous solution, and the point $B$ will be without the limiting parabola. If the circles touch, one parabola can be drawn, having its focus upon the line $AB$, the point $B$ being then upon the limiting parabola; while if the circles intersect, there will be two parabolic paths along which the particle may move, the first having its focus below, and the second above the line $AB$, the point $B$ being in this case within the limiting parabola.

**215.** It will be seen that by changing the independent variable we avoid any infinite value of $y'$, and we will now proceed to show that when the parabolic arc has its focus below the line $AB$, the action becomes a minimum, but that when the focus is upon or above $AB$, the action is not a minimum.

Employing Jacobi's method, we have, from (9),

$$\frac{d^2V}{dy'^2} \quad \text{or} \quad a_{y'y'} = \frac{\sqrt{y}}{\sqrt{(1+y'^2)^3}}, \tag{13}$$

which is always positive, and remains finite throughout the range of integration, so that we shall have a minimum if we can take $u$ so that it shall not vanish within the same range, and that $u'$ may remain finite. From (10) we have

$$\frac{dy}{dC_1} = 1 - \frac{(x-C_2)^2}{4C_1^2} = 1 - y'^2, \qquad \frac{dy}{dC_2} = -\frac{x-C_2}{2C_1} = -y', \tag{14}$$

the value of $y'$ being taken from (11). Therefore the most general value of $u$ is

$$u = 1 - y'^2 - Ly', \qquad \text{and} \quad u' = -2y'y'' - Ly''. \tag{15}$$

Now because $y'$ and $y''$ remain finite, $u'$ will not become infinite; and to make $u$ vanish, we must have

$$L = \frac{1}{y'} - y' = -H, \tag{16}$$

and we shall have a minimum if the range of $H$ over all real values be only partial, but none if it be complete.

**216.** In order that $H$ may range over all real values, it must certainly touch zero and infinity. The first condition requires $y'$ to become $\pm 1$, and is fulfilled at both extremities of the latus rectum, and there only. The second requires $y'$ to become either zero or infinity, the latter condition being never fulfilled, and the former at the vertex only. Now let $y_1'$ and $y_4'$ be the values of $y'$ at the extremities of any focal chord. Then, because the tangents to the parabola at these extremities meet at right angles upon the directrix, we must have $y_1' = -\frac{1}{y_4'}$, and hence we shall find that

$$y_1' - \frac{1}{y_1'} \quad \text{or} \quad H_1 = y_4' - \frac{1}{y_4'} \quad \text{or} \quad H_4.$$

Therefore as $H$ in this case starts with a certain value, changes sign by passing through infinity at the vertex, and returns to its initial value, its range must be at least complete, and we have not a minimum. If the arc were still greater, the range of $H$ would be more than complete.

Now $H$ can return to its initial value but once, although it may pass that value. When the initial value is zero, this is evident, since, as we have seen, there is but one other point at which $H$ can be zero. When the initial value is not zero, $H$ must change sign twice before returning to its initial value, and four times before returning to it a second time, and this latter is impossible, since there are but three points at which $H$ can change sign at all.

Since, then, the values of $H$ at the extremities of any focal chord are equal, they will be equal nowhere else, and the range of $H$ is then just complete.

If, therefore, the arc in question be less than that subtended by a focal chord, the range of $H$ is not complete, and the action becomes a minimum. In other words, we see that the action will not be a minimum unless the second fixed point be so situated that two parabolic arcs are admissible, and then for that path only which has its focus below the line $AB$.

**217.** Since there can be no continuous solution when the second fixed point lies on or without the limiting parabola, we next inquire whether there may not be some discontinuous solution or solutions in these cases.

We first ask, then, whether we have unconsciously imposed any boundary along which the sign of $\delta y$ is fettered; because, if not, the solution can, at least so far as discoverable by the calculus of variations, consist only of some combination of lines satisfying the equation $M = 0$. But we see from (10) that $y = 0$ is such a boundary, since to make $y$ negative would render the velocity imaginary, and with the notation of (1) this boundary is given by the equation $x + a = 0$.

**218.** Let us next see what can be obtained from the fundamental equations given by the two methods previously employed. These are

$$\sqrt{x+a}\,\frac{y'}{\sqrt{1+y'^2}} = \sqrt{c}, \qquad \frac{y}{1+y'^2} = C_1, \tag{17}$$

because there is no escape from these, if we make $M$ vanish in each case. The first of these equations is satisfied by $y' = 0$ and $c = 0$, and also $y' = \infty$, because in the latter case we obtain $x + a = c$.

Passing for the present the question of combining $y' = 0$ and $y' = \infty$, it is suggested that our solution may consist, in

part, of some line parallel to the axis of $y$. But because $y'$ would here become infinite, we cannot, while keeping the vertical as the axis of $x$, investigate the variation of $U$ along this line. But the second of equations (17), in which the axis of $x$ is horizontal, offers the same solutions as the first, since it is satisfied by $y' = \infty$, $C_1 = 0$, or by $y' = 0$, which gives $y = C_1$, which is the same condition as was before expressed by $x + a = 0$.

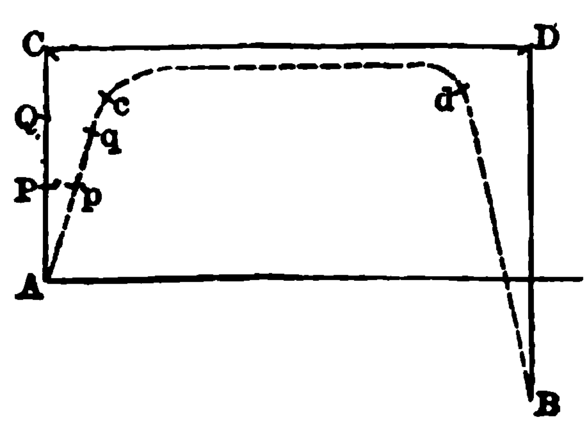
With this change of the independent variable, we can examine the condition which we were before unable to investigate; namely, whether the solution may be composed, in part, of some horizontal line.

Now if this horizontal be any other than the boundary $y = 0$, it must, since along it $\delta y$ is of unrestricted sign, satisfy the equation $M = 0$. But this equation, when $U$ has the form given in (10), becomes

$$M = \frac{\sqrt{1 + y'^2}}{2\sqrt{y}} - \frac{d}{dx}\frac{y'\sqrt{y}}{\sqrt{1 + y'^2}}. \tag{18}$$

But when we put $y' = 0$ and $y = C_1$, we have $M = \frac{1}{2\sqrt{C_1}}$: and as this does not vanish, this solution must be abandoned.

**219.** As the horizontal line $y = 0$ is not yet known to be excluded, since it need not satisfy the equation $M = 0$, and as




$y' = \infty$ was also suggested as a solution, it remains to consider whether the solution may not be found by combining this hori-

zontal with the verticals through the two fixed points, as in the figure, where the path of the projectile is supposed to be $ACDB$.

Of course a particle could not move from $A$ to $B$ along this broken line, because its velocity along $CD$ would become zero. But we can draw a curve indefinitely near to $ACDB$ along which the velocity will not become exactly zero, and then we shall find that the action along this curve will be greater than that along the discontinuous path.

To determine whether the line $ACDB$ is the path of minimum action, we shall, on account of the infinite value of $y'$, need some other method of investigation, and we might try transforming to polar co-ordinates. Still an analytical demonstration will not here be necessary. For let $AC$ and $Ac$ be equal in length, and let them be divided into the same number of equal and infinitesimal parts; and let $PQ$ and $pq$ be a corresponding pair, so that $AP$ will equal $Ap$. Then because $P$ is vertically higher than $p$, the velocity at $P$ will be less than that at $p$, and the action through $PQ$ less than that through $pq$. Hence it appears that the entire action through $AC$ is less than that through $Ac$. In like manner we show that the action through $BD$ is less than that through $Bd$. Now the action along $CD$ is zero, while that along $cd$ is not; so that it is certain that the action along the primitive $ACDB$ is less than that along the derivative $AcdB$, even if we do not vary the boundary $CD$, and much more so if we vary that line.

**220.** It is easy to see that the discontinuous solution which we have obtained is admissible even when the parabolic path also renders the action a minimum. When the second fixed point lies on or without the limiting parabola, the discontinuous solution, being the only one which presents itself, undoubtedly renders the action the least possible, as well as a minimum. When both minima are admissible, we shall find that sometimes the one and sometimes the other will give the smaller

minimum; and there can be little doubt this smaller minimum is in every case the least possible value also of the action.

The comparison of the two minima, when they exist, must be effected by the ordinary calculus, but we subjoin, without proof, the necessary formulæ. (See Todhunter's Researches, Art. 173.)

Let $g$ be the force of gravity, $r_0$ and $r_1$ the radii vectores of the two fixed points, $C$ the length of the chord joining these points, and $w$ the action. Then for the parabolic path, according as it subtends less or more than two right angles at the focus, we shall have

$$\left.\begin{aligned} w &= \frac{\sqrt{g}}{3}\{(r_0+r_1+C)^{\frac{3}{2}} - (r_0+r_1-C)^{\frac{3}{2}}\}, \\ &\text{or} \\ w &= \frac{\sqrt{g}}{3}\{(r_0+r_1+C)^{\frac{3}{2}} + (r_0+r_1-C)^{\frac{3}{2}}\}. \end{aligned}\right\} \qquad (19)$$

For the discontinuous solution the action is that due to passing along the verticals only, and is

$$w = \frac{2\sqrt{2g}}{3}(r_0+r_1). \qquad (20)$$

**221.** The principles which have been previously explained regarding the origin and nature of discontinuous solutions are equally applicable when polar co-ordinates are employed, and we shall find in this case also that they are generally in some manner presented as a solution of the equation $M=0$, although they may not, and need not always, really satisfy that equation at all. Let us now briefly consider a problem of this kind.

## Problem XXXVI.

*It is required to determine whether there be any discontinuous solution involved in Prob. XXII.*

We have seen, Art. 123, that when the second fixed point lies without a certain limiting ellipse, no elliptic arc, satisfying all the conditions of the question, can be drawn between it and the first fixed point; and that even when it is situated on the limiting ellipse, and there can be one ellipse drawn, it does not render the action a minimum. It appears, then, that if there be any solution in these cases, it must be discontinuous; and the analogy of the last problem would lead us to expect, what is indeed the fact, that even when a continuous solution exists, a still smaller value of the action is in some cases given by a certain discontinuous solution.

**222.** Now the fundamental equation of this problem is equation (8),

$$\frac{Wr^2}{\sqrt{r^2+r'^2}} = c, \tag{1}$$

where

$$W = \sqrt{\frac{2}{r} - \frac{1}{a}}, \tag{2}$$

and we cannot help arriving at this equation if we make $M$ vanish. But if in (1) we make $c$ zero, that equation will be satisfied by $r' = \infty$ or $W = 0$. The first would indicate that we might employ some portion of the radius vector drawn to one or both the fixed points, or of these radii produced.

To interpret the second we have

$$W^2 = \frac{2}{r} - \frac{1}{a} = 0,$$

so that

$$\frac{1}{r} = \frac{1}{2a}, \qquad r = 2a.$$

That is, it is suggested that a portion of the solution might consist of a circular arc described from the centre of force with a radius $2a$.

Let $O$ be the centre of force, $A$ and $B$ the two fixed points. Then the discontinuous solution which is proposed is the path $ACDB$, where $CD$ is the portion of the above-named circular arc intercepted between $OA$ and $OB$ produced.

But before considering whether the proposed solution does render the action a minimum, we inquire whether any boundary exists along which the sign of $\delta r$ is fettered, and which need not therefore satisfy the equation $M = 0$. Now the value of the velocity $v'$, equation (4), Art. 121, is

$$v' = \sqrt{\frac{2f}{r} - \frac{f}{a}} = W\sqrt{f}, \tag{3}$$

where $f$ is the intensity of the attracting force at a unit's distance. When, therefore, $W$ vanishes and $r$ becomes $2a$, $v'$ becomes zero; and when we make $r$ greater than $2a$, $v'$ becomes imaginary. Hence the arc $CD$ is itself such a boundary, unconsciously imposed, and along it $\delta r$ must be negative and the action zero.

**223.** Owing to the infinite value of $r'$, we cannot determine, by adhering to polar co-ordinates, whether the proposed solution will render the action a minimum or not, and the natural mode of procedure would be to express the value of $\delta U$ in rectangular co-ordinates, by which we could escape infinite values. But this will not be necessary, because, by reasoning precisely similar to that employed in Art. 219, it will appear that the action through $AC$ and $DB$ must be less than that through any curve of the same length which can be derived by the method of variations, and the arc $CD$ cannot be reached by curves which do not exceed these lines. Then as the action is zero along $CD$, it is evident that the discon-

tinuous path $ACDB$ will render the action less than would any other path which could be derived from it by the calculus of variations.

### Problem XXXVII.

**224.** *A steamer is to pass from one port to another on a stream whose current flows always in the same direction, her speed being dependent solely upon the angle which her course makes with the direction of the current, together with certain constant quantities. It is required to determine the form of her path, so that the passage may be made in the shortest time possible.*

Assume the course of the current as the axis of $x$, and estimate $x$ in the direction of its flow. Also let $v$ be the velocity, $t$ the time, and $ds$ an element of the required path. Then since $v$ depends, in some fixed manner, upon constants and the angle between the path and the axis of $x$, we may write $v = F(y')$, and

$$dt = \frac{ds}{v} = \frac{\sqrt{1 + y'^2}\, dx}{F} = f(y')\, dx = f dx.$$

Hence the expression to be minimized is $U = \int_{x_0}^{x_1} f dx$, where it is evident that $f$ can become any function whatever of $y'$.

Now we have already seen, Art. 56, that the solution of this problem is given always by a straight line, and there is no escape from this conclusion so long as we make $M$ vanish. For

$$M = -\frac{d}{dx}\frac{df}{dy'} = -\frac{df'}{dx},$$

so that if $M$ be zero, we cannot help obtaining $f' = c$; and to satisfy this equation, $y'$ must certainly be a constant, which will lead to a right line as the only possible solution. But since the required line is in this case to pass through two fixed

points, we seem at first to be restricted to a single course for all possible conditions, whereas a little reflection will serve to show us that we could easily impose such conditions as would enable us to shorten the time of passage by pursuing a path not always coinciding with the straight line joining the two points.

It appears, however, upon examination, that the equation $M = 0$ must hold throughout the entire course, as we cannot find that any boundary has been in any way imposed along which $\delta y$ or $\delta y'$ will be of restricted sign. We feel certain, therefore, that no solution can be obtained, at least by the calculus of variations, except a right line, or one composed of right lines. But since $f'$ is a constant, suppose that constant to become zero. Then if the equation $f' = 0$ furnish more than one real value of $y'$, we may have two or more lines meeting at finite angles. For the terms free from the sign of integration, which are $f_2'\delta y_2 - f_1'\delta y_1 +$ etc., would vanish, because $f'$ would vanish for either of the meeting lines, although the values of $y'$ for the two lines might differ. We shall, however, illustrate this problem by considering some particular cases.

**225.** 1st. Let $a$ be the angle between the path and the axis of $x$, which is not to exceed $\frac{\pi}{2}$, and suppose the velocity $v$ to vary as $\cos a = \frac{1}{\sec a} = \frac{1}{\sqrt{1+y'^2}}$. Then in this case $U$ becomes $U = \int_{x_0}^{x_1} (1 + y'^2)dx = \int_{x_0}^{x_1} f dx$, giving $f' = \frac{df}{dy'} = 2y'$. Now $y'$ must have the same value throughout the integral, because if it change value at any point $x_2, y_2$, we shall have, as already explained, without the integral sign, after transforming the term of the first order in the usual way,

$$f_2'\delta y_2 - f_3'\delta y_2, \quad \text{or} \quad 2(y_2'\delta y_2 - y_3'\delta y_2), \quad \text{or} \quad 2(y_2' - y_3')\delta y_2,$$

which must vanish, since $\delta y_2$ may have either sign. Hence, in this case, the minimum time will be gained by following the right line joining the two points; and because only one value of $y'$ is admissible, we infer that this path gives also the least value of $t$, $t$ being certainly a minimum, since the term of the second order is $\int_{x_0}^{x_1} \delta y'^2 dx$. In this case, then, there is no discontinuity, but we now pass to an example in which it occurs.

**226.** 2nd. Let

$$v = \frac{\sqrt{1 + y'^2}}{b^2 - \frac{y'^2}{2} + \frac{y'^4}{4}},$$

so that

$$f = b^2 - \frac{y'^2}{2} + \frac{y'^4}{4}, \tag{1}$$

where $b$ is some constant. Then proceeding as usual with the integral $U = \int_{x_0}^{x_1} f dx$, we obtain

$$f' = y'(y'^2 - 1) = c. \tag{2}$$

Whence we also find

$$\frac{d^2 f}{dy'^2} = f'' = 3y'^2 - 1. \tag{3}$$

Now if we solve (2) without restriction, we shall obtain a straight line, which must of course pass through the two fixed points, and we will first examine whether this continuous solution will always render the time of passage a minimum. Now since the term of the second order in $\delta U$ is

$$\frac{1}{2}\int_{x_0}^{x_1} f'' \delta y'^2 dx,$$

we shall have a maximum or a minimum according as $f''$ is negative or positive. Hence, from (3), observing that $\tan^2 \frac{\pi}{6} = \frac{1}{3}$, we see that when the angle is less than $\frac{\pi}{6}$, the time is a maximum; but that if the ports were so situated that the line joining them must make with the axis of $x$ an angle greater than $30°$, the time will become a minimum.

**227.** Now when we have shown $t$ to be in any particular case a maximum or a minimum, it does not follow that we have obtained its greatest or least value, since some discontinuous solution may give a greater maximum or a smaller minimum. Now if there be any discontinuous solution, it must cause $f'$ or $c$ in (2) to retain the same value throughout $U$, otherwise there would arise terms of the form $(f_2'-f_1')\,\delta y_1$, which would not vanish. Any values, then, of $y'$ which will satisfy the equation $f'=c$, in which we may give to $c$ any value we please, only retaining the same throughout $U$, may be combined into one solution, provided this combination will enable us to pass from one fixed point to the other, and provided also that the various parts of the combination do not render the terms of the second order of variable or conflicting sign.

Suppose, in the present case, we make $c$ zero. Then we obtain, as the roots of (2), $y'=0$, $y'=1$, $y'=-1$. But the last two values of $y'$ render $f''$ in (3) positive, while the first renders it negative, and cannot, therefore, enter any solution with the other two, as the sign of the terms of the second order would then be in our power. It is evident that a vessel could pass from one point to any other by a suitable combination of tacks, making with the axis of $x$ angles whose tangents are either $+1$ or $-1$; and as the integral has the same value, whatever be the number of these tacks, because $f$ is the same whether $y'$ be $+1$ or $-1$, we obtain in all cases one path along which the time of passage will be a minimum.

To determine, when two minima exist, whether the quicker passage can be made by following the path composed of tacks or a continuous line, is not a problem of variations, but of algebra only. For, resuming the value of $f$, we may write

$$f = b^2 - \frac{y'^2}{2} + \frac{y'^4}{4} = b^2 - \frac{1}{4} + \frac{(y'^2 - 1)^2}{4}.$$

Also, when $y'$ is $+ 1$ or $- 1$, $f = b^2 - \frac{1}{4}$. Therefore, since $(y'^2 - 1)^2$ cannot become negative, we see that the solution composed of tacks will give the least possible value of $t$. We have, of course, assumed that it is not necessary to tack backward; that is, that $x$ may always increase algebraically.

**228.** We naturally inquire whence arises the discontinuity in this class of problems, and why it presents itself in certain forms of $f$, and not in others. Now the only condition imposed besides the fundamental one, that the given line shall possess a certain maximum or minimum property, is that it shall also join two fixed points, and if the required maximum or minimum property be not altogether impossible, the discontinuity must result from imposing the second condition. That it does in general thus arise will appear from the following example, in which this condition is removed.

## Problem XXXVIII.

**229.** *A vessel starting from a fixed point is required to sail a certain number of miles, her speed being always dependent solely upon the direction of her course and certain constant quantities. It is required to determine along what path the given distance may be accomplished in a minimum time.*

Regarding the ocean as a plane, assume the meridian through the starting-point as the axis of $x$, and employ

$t$ and $v$ as before. Then it is plain that we shall have, as formerly, $v = F(y') = F$, and $dt = \frac{\sqrt{1+y'^2}}{F} dx = f(y')\,dx = fdx$. Hence we are to minimize the expression $\int_{x_0}^{x_1} fdx$, while $\int_{x_0}^{x_1} \sqrt{1+y'^2}\,dx$ is to remain constant. Therefore the problem is now one of relative maxima and minima, and we have

$$U = \int_{x_0}^{x_1} \left(f + a\sqrt{1+y'^2}\right) dx = \int_{x_0}^{x_1} Vdx, \tag{1}$$

where it must be observed that $V$ is also a function of $y'$ and constants only. In the present case, moreover, we do not suppose the second extremity of the required curve to be in any manner restricted, so that $x_1$ and $y_1$ are both variable. Therefore, to the first order, we have

$$\delta U = \left(f + a\sqrt{1+y'^2}\right)_1 dx_1 + \left\{f' + \frac{ay'}{\sqrt{1+y'^2}}\right\}_1 \delta y_1$$

$$- \int_{x_0}^{x_1} \frac{d}{dx}\left\{f' + \frac{ay'}{\sqrt{1+y'^2}}\right\} \delta y\, dx. \tag{2}$$

Whence

$$f' + \frac{ay'}{\sqrt{1+y'^2}} = c, \tag{3}$$

where

$$f' = \frac{df}{dy'}.$$

Now, for the same reason as given in the preceding problem, $c$ cannot, even should discontinuity occur, and the integral be separated into parts, have two values, $c_1$ and $c_2$, within the range of integration; and since we know from (3) that $c$, or the coefficient of $\delta y_1$, must vanish, (3) becomes

$$f' + \frac{ay'}{\sqrt{1+y'^2}} = 0. \tag{4}$$

Moreover, in this case, no relation exists between $dx_1$ and $\delta y_1$, because the extremity of the required curve is not confined to any other curve, but is wholly unrestricted. Therefore (1) must also give

$$(f + a\sqrt{1+y'^2})_1 = 0. \tag{5}$$

From (4) and (5) we have

$$a = \left\{\frac{-f}{\sqrt{1+y'^2}}\right\}_1 \quad \text{and} \quad a = \frac{-f'\sqrt{1+y'^2}}{y'}. \tag{6}$$

From (4) and (6) we obtain

$$\left\{f' - \frac{y'f}{1+y'^2}\right\}_1 = 0. \tag{7}$$

Now since $V$ is a function of $y'$ only, we know that the required path must be some right line, or combination of right lines, so that $y'$ is the tangent of the inclination of this line, or else of the last tack, to the axis of $x$. But it is evident that if the solution can consist of tacks, involving two or more values of $y'$, the arrangement of these tacks will be arbitrary, since the integral taken through any given portion of $x$ will be the same for any one of the tacks—that is, for any one of the admissible values of $y'$—and therefore $y_1'$ can have any one of these values, but no others. Hence, as the possible values of $y'$ and $y_1'$ are the same, we may remove the suffix from (7) and write, as the general equation of condition,

$$f' - \frac{y'f}{1+y'^2} = 0. \tag{8}$$

From (8) we can obtain $y'$ in terms of constants only, and it may have one or more real values, the imaginary roots being of course rejected. In the former case there can be but one solution; but when $y'$ has more than one real value, a discon-

tinuous solution by a combination of these values would seem possible. It must, however, be observed that, whether the solution be continuous or not, $a$ must retain the same value throughout $U$.

Now take any two real values of $y'$ found from (8), and make $y_1'$ equal to the first, and $y_2'$, which may be regarded as measuring the slope of some other tack, equal to the second. Then from (6), and also observing that we may interchange the slopes of the tacks at pleasure, we have

$$\left.\begin{aligned} \left\{\frac{f}{\sqrt{1+y'^2}}\right\}_1 &= \left\{\frac{f'\sqrt{1+y'^2}}{y'}\right\}_2, \\ \left\{\frac{f}{\sqrt{1+y'^2}}\right\}_2 &= \left\{\frac{f'\sqrt{1+y'^2}}{y'}\right\}_1; \end{aligned}\right\} \qquad (9)$$

and as every member in (9) equals $-a$, we may write

$$\begin{aligned} \left\{\frac{f}{\sqrt{1+y'^2}}\right\}_1 &= \left\{\frac{f}{\sqrt{1+y'^2}}\right\}_2 \\ = \left\{\frac{f'\sqrt{1+y'^2}}{y'}\right\}_1 &= \left\{\frac{f'\sqrt{1+y'^2}}{y'}\right\}_2. \end{aligned} \qquad (10)$$

But it will be in general impossible to satisfy (10) by employing any two values of $y'$ found from (8), so that a discontinuous solution will not frequently occur. Still such solutions are possible, as we shall presently show; and even when no discontinuity is admissible, it is conceivable that we may have a choice of two continuous solutions, provided $y_1'$ and $y_2'$ can severally satisfy the equations

$$\left\{\frac{f}{\sqrt{1-y'^2}}\right\}_1 = \left\{\frac{f'\sqrt{1+y'^2}}{y'}\right\}_1,$$

$$\left\{\frac{f}{\sqrt{1+y'^2}}\right\}_2 = \left\{\frac{f'\sqrt{1+y'^2}}{y'}\right\}_2,$$

because $a$ in the two solutions need not be identical, but must not change value in the same solution.

**230.** As a particular example of the preceding problem, let us assume the velocity to be that employed in case 2nd, Prob. XXXVII., so that $f$ and $f'$ will have the values there given. Then by equations (1) and (2), Art. 226, equation (8) of the preceding article will become

$$y'(y'^2 - 1) - y' \frac{b^2 - \frac{y'^2}{2} + \frac{y'^4}{4}}{1 + y'^2} = 0. \tag{1}$$

Therefore, if $y'$ be not zero, we have

$$y'^4 - 1 - \left\{ b^2 - \frac{y'^2}{2} + \frac{y'^4}{4} \right\} = 0$$

or

$$y'^4 + \frac{2y'^2}{3} = \frac{4(b^2 + 1)}{3}. \tag{2}$$

Whence

$$\left.\begin{aligned} y'^2 &= -\frac{1}{3} \pm \sqrt{\frac{4}{3}\left(b^2 + \frac{13}{12}\right)} = -\frac{1}{3} \pm \sqrt{B}, \\ y' &= \pm \sqrt{-\frac{1}{3} \pm \sqrt{B}}. \end{aligned}\right\} \tag{3}$$

Now if $\sqrt{B}$ be less than $\frac{1}{3}$, $y'$ will always be imaginary; if it equal $\frac{1}{3}$, $y'$ will be zero; while if it exceed $\frac{1}{3}$, one of the values of $y'^2$ will be negative, and all the real values of $y'$ will be given by the equation

$$y' = \pm \sqrt{-\frac{1}{3} + \sqrt{B}}. \tag{4}$$

Now we have

and

$$\left.\begin{aligned}\frac{f}{\sqrt{1+y'^2}} &= \frac{b^2-\frac{y'^2}{2}+\frac{y'^4}{4}}{\sqrt{1+y'^2}}\\ \frac{f'\sqrt{1+y'^2}}{y'} &= (y'^2-1)\sqrt{1+y'^2}.\end{aligned}\right\} \qquad (5)$$

But it at once appears that none of the members of (9) will be in any way affected by the successive substitution of two values of $y'$ numerically equal but with contrary sign. Moreover, in this case, the equation

$$\frac{f}{\sqrt{1+y'^2}} = \frac{f'\sqrt{1+y'^2}}{y'}$$

reduces to equation (2); so that it must be satisfied by either value of $y'$ just found, and will also, from what has been shown, be satisfied by substituting the positive value in one member and the equal negative value in the other.

Hence it appears that equation (10) of the preceding article will be satisfied by putting for $y_1'$ and $y_2'$ the two values of $y'$ given in (4), and by no others. Therefore the solution $y'=0$ can only hold when zero is a root of (2), which can only be made true by making $b^2=-1$, and in this case there will be no other root, and so no discontinuity. But if $b^2$ become greater than $-1$, we shall have an equal positive and negative value of $y'$, which may be combined in the same solution, thus giving discontinuity.

**231.** Let us now consider the terms of the second order. These are

$$\delta U = \frac{1}{2}\left(\frac{dV}{dx}\right)_1 dx_1^2 + \delta V_1\, dx_1 + \frac{1}{2}\int_{x_0}^{x_1} \frac{d^2V}{dy'^2}\,\delta y'^2\,dx, \qquad (6)$$

where

$$V = f + a\sqrt{1+y'^2} = b^2 - \frac{y'^2}{2} + \frac{y'^4}{4} + a\sqrt{1+y'^2}.$$

Now, because $y''$ is zero, all the successive differential coefficients of $V_1$ must vanish, and also we have

$$\delta V_1 = \left\{\frac{dV}{dy'}\right\}_1 \delta y_1' = \left\{f' + \frac{ay'}{\sqrt{1+y'^2}}\right\}_1 \delta y_1'$$

$$= \left\{y'(y'^2 - 1) + \frac{ay'}{\sqrt{1+y'^2}}\right\}_1 \delta y_1';$$

and, as will be seen from equation (4), Art. 229, the coefficient of $\delta y_1'$ likewise vanishes, so that we have left in $\delta U$ only the terms under the integral sign, and have merely to determine the sign of $\frac{d^2V}{dy'^2}$. Now we have

$$\frac{d^2V}{dy'^2} = f'' + \frac{a}{\sqrt{(1+y'^2)^3}}, \qquad f'' = 3y'^2 - 1,$$

$$a = \frac{-f'\sqrt{1+y'^2}}{y'} = -(y'^2 - 1)\sqrt{1+y'^2}.$$

Whence

$$\frac{d^2V}{dy'^2} = 3y'^2 - 1 - \frac{y'^2 - 1}{1+y'^2} = \frac{3y'^4 + y'^2}{1+y'^2}.$$

Therefore it appears that we have a minimum whether $y'$ be positive or negative.

When, however, $y'$ is zero, we see from the last equation that $\frac{d^2V}{dy'^2}$ is also zero, so that we might infer that this value of $y'$ gives neither a maximum nor a minimum. But this inference would not in the present case be correct, because we shall find that the terms of the third order reduce also to zero, while those of the fourth order will become positive. It may be also observed, although not affecting the problem, that

when $y'$ is zero, $dx_1$ can have but one sign, the negative, if $x_1$ be positive.

**232.** It will be seen that while the removal of all conditions regarding the upper limit does not here destroy the admissibility of a discontinuous solution, it nevertheless abolishes its necessity. For as the value of $f$, and also that of $x_1$, will be the same whether we employ the positive or the negative value of $y'$, or some combination of the two, the time $\int_{x_0}^{x_1} fdx$ will be also unaltered; and as we are not now obliged to tack in order to go from one fixed point to another, and no time is gained by tacking, the discontinuity is merely admissible. The discontinuity in this case appears to arise from the fact that the problem is so constructed that the fundamental equation $f' + \frac{ay'}{\sqrt{1+y'^2}} = c$ may have two roots, both of which give the same value of $t$, and satisfy all the conditions of the question.

**233.** Suppose we modify the preceding example by requiring that, instead of sailing a certain number of miles, the vessel shall be required to reach a certain degree of latitude in a minimum time. Then we are to minimize absolutely the expression $U = \int_{x_0}^{x_1} fdx$, where $f$ has the same value as before, the limit $x_1$ now being fixed, but $y_1$ being subject to variation. Then we have, as before, $f' = c$, and $c$ cannot have two values. But because $\delta y_1$ is not zero, $f_1'$ or $f'$ must vanish, so that we have $f' = y'(y'^2 - 1) = 0$; the roots of which are $y' = 0$, $y' = 1$, $y' = -1$. Now as $y_1'$ is not fixed, we can employ any one of the values of $y'$ alone throughout $U$. The first will render $U$ a maximum, as we have already seen, while the other two will give $U$ the same value whether employed separately or in combination, which value is a minimum, as has been shown, and is also the least value of $U$.

**234.** We may now consider briefly the inquiry with which we opened Art. 228.

Two things affect the problem: first, the particular form of $f$ or $V$; and second, the conditions which are to hold at the limits. With regard to the first we may observe that there can be no discontinuity unless $f$ or $V$ be of such a form that the fundamental equation $f' = c$ can furnish more than one real value of $y'$. Thus, in Prob. I., the fundamental equation is $\frac{y'}{\sqrt{1+y'^2}} = c$, which, because $\sqrt{1+y'^2}$ is supposed to remain positive, can be satisfied by one value of $y'$ only, so that in this case no discontinuity is possible.

Second, when the fundamental equation gives several real values of $y'$, and a combination of them satisfies all the other conditions of the question, the necessity for the employment of this combination, or discontinuous solution, generally arises from the fact that the points to be joined are fixed. Moreover, as we in whole or in part remove this restriction from one of the limits, we decrease the probability that these values can be combined at all; that is, that discontinuity will be possible; and even when it still occurs, it appears generally rather admissible than necessary.

**235.** When $f$ is a function of $y''$ or $y'''$ only, admissible, but not necessary discontinuity is still more likely to occur. Let us consider, as an illustration, a particular case of Prob. IV.

## Problem XXXIX.

*Let it be required to maximize or minimize the expression*

$$U = \int_{x_0}^{x_1} \left\{ a^4 y''^2 + \frac{b^4}{y''^2} \right\} dx = \int_{x_0}^{x_1} f(y'')\, dx = \int_{x_0}^{x_1} f dx, \qquad (1)$$

*supposing the limiting values of $x$ and $y$ only to be fixed.*

Proceeding as usual, we obtain

$$f' = \frac{df}{dy''} = 2\left(a^4 y'' - \frac{b^4}{y''^3}\right) = c_1 x + c_2. \tag{2}$$

But $\delta y_1'$ and $\delta y_0'$ are not zero, so that their coefficients $f_1'$ and $f_0'$ must severally vanish; and assuming the origin at one of the fixed points, we readily see that $c_1$ and $c_2$ also vanish, so that (2) gives

$$a^4 y'' - \frac{b^4}{y''^3} = 0 \tag{3}$$

and

$$y'' = \pm\frac{b}{a} = \pm B. \tag{4}$$

Therefore, by integration, we obtain

$$y = \pm\frac{Bx^2}{2} + C_1 x + C_2, \tag{5}$$

in which, because the origin is at one of the fixed points, $C_2$ must vanish, and then $C_1$ must be determined by making the parabola pass through the second fixed point, whose co-ordinates must satisfy the equation

$$y_1 = \pm\frac{bx_1^2}{2a} + C_1 x_1.$$

The term of the second order is

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\frac{d^2 f}{dy''^2}\,\delta y''^2 dx = \int_{x_0}^{x_1}\left\{a^4 + \frac{3b^4}{y''^4}\right\}\delta y''^2 dx, \tag{6}$$

which is positive for either value of $y''$, thus giving a minimum.

We have here also the least value of $U$. For we may write

$$a^4y''^2 + \frac{b^4}{y''^2} = \left(a^2y'' - \frac{b^2}{y''}\right)^2 + 2a^2b^2, \tag{7}$$

which, by making $y''$ either $+\frac{b}{a}$ or $-\frac{b}{a}$, reduces to $2a^2b^2$.

**236.** Here no discontinuous solution can be necessary, because we can always join the two fixed points by a parabolic arc, in which $y''$ shall be $+\frac{b}{a}$ or $-\frac{b}{a}$; and also, we have then the least value of $U$. Still, a discontinuous solution is always admissible. For we can also always pass from the first to the second fixed point by some combination of parabolic arcs, each of which will satisfy (5), but will differ in the values of $C_1$ and $C_2$.

Now it is evident that all these arcs, having $y''$ either $+\frac{b}{a}$ or $-\frac{b}{a}$, will satisfy the equation $M = 0$, and it remains only to show that they will also make the terms in $\delta U$ which remain without the integral sign vanish.

Consider two of these arcs meeting at the point $x_2, y_2$. The terms arising for this point are

$$-\left\{\left(\frac{df'}{dx}\right)_2 - \left(\frac{df'}{dx}\right)_2\right\}\delta y_2 + f_2'\,\delta y_2' - f_2'\,\delta y_2'.$$

But since the equation $y'' = \pm\frac{b}{a}$ holds for both arcs, $f'$ and $\frac{df'}{dx}$ must vanish for both, thus rendering the expression just given likewise zero; and similarly for any number of arcs.

Here the discontinuous solution consists of parabolic arcs which may meet at finite angles, and the value of $U$, and also that of the terms of the second order, is the same for either solution.

## Problem XL.

**237.** *It is required to determine the solution of Prob. XV. when the length of the given line exceeds that of the semi-circumference described upon the line joining the two fixed points as a diameter.*

We can of course always, by taking the radius sufficiently great, join two points by a circular arc, whatever the length of that arc may be required to be. But we cannot here extend the arc beyond 180°; because then there would be beyond $y_0$ and $y_1$ both a convex and a concave portion of the arc; and besides being compelled to count a portion of the area twice, these portions would, as we have seen in Art. 95, give opposite signs to the terms of the second order. Indeed, whatever may be the solution, we would most naturally understand the problem to imply that we are not to go beyond the production of the ordinates $y_0$ and $y_1$; that is, beyond the lines whose equations are $x = x_0$ and $x = x_1$, which may therefore be considered as boundaries which we must not transgress.

We would therefore feel certain that the solution, at least so far as discoverable by the calculus of variations, can consist only of what will satisfy the equation $M = 0$, with perhaps some portion of these boundaries, unless indeed some other boundary can be discovered.

Let us now see what can be obtained in the usual way. We have

$$\left.\begin{aligned} U = \int_{x_0}^{x_1}\left(y + a\sqrt{1+y'^2}\right)dx = \int_{x_0}^{x_1} V dx, \\ M = 1 - \frac{d}{dx}\frac{ay'}{\sqrt{1+y'^2}}, \quad \text{and} \quad x - \frac{ay'}{\sqrt{1+y'^2}} = c. \end{aligned}\right\} \quad (1)$$

Now the last equation will be satisfied by $y' = \infty$, because we shall then obtain $x - a = c$, which is therefore a particular or

singular solution, being the equation of a right line perpendicular to $x$. But any such line will reduce $M$ to unity, so that we can only employ one or both boundaries joined to a circular arc, because that arc gives the only general solution of the equation $M = 0$.

Moreover, we cannot assert that $c$ must retain in this case the same value throughout $U$. For the terms without the integral sign at either point of junction of the arc and line are of the general form

$$a\left\{\left(\frac{y'}{\sqrt{1+y'^2}}\right)_2 - \left(\frac{y'}{\sqrt{1+y'^2}}\right)_3\right\}\delta y_2, \qquad (2)$$

which in order to vanish will require that $y'$ shall at these points mean the same thing for the arc and the line; that is, that they shall be tangent. Hence we are not confined to one boundary, but are at liberty to employ both.

**238.** As the infinite values of $y'$ will render our investigations untrustworthy, we must, in order to determine whether the proposed combination be the real solution, transform to polar co-ordinates. Take the pole at any point on the axis of $x$, between $x_0$ and $x_1$, regarding that axis as the initial line, and denoting by $v$ the angle which any radius vector $r$ makes with this initial. Then it is plain that $U$ must have the general form given in equation (3), Prob. XXIII., except that the limits will not be the same. For let $v_0$ and $v_1$ be the respective angles which the radii $r_0$ and $r_1$ drawn to the two fixed points make with the initial. Then we need only consider the integral from $v_0$ to $v_1$, because although all the area in question is not comprised within the limits, still the two remaining triangles which are included between the initial and the respective radii and ordinates undergo no variations.

We are, then, to maximize the expression

$$U = \int_{v_0}^{v_1}\left\{\frac{r^2}{2} + a\sqrt{r^2 + r'^2}\right\}dv = \int_{v_0}^{v_1} V dv. \qquad (3)$$

Then, since $\delta r_0$ and $\delta r_1$ vanish, if we suppose $U$ divided as our solution requires, we shall have

$$\delta U = a\left\{\left(\frac{r'}{\sqrt{r^2+r'^2}}\right)_2 - \left(\frac{r'}{\sqrt{r^2+r'^2}}\right)_3\right\}\delta r_2$$

$$+ a\left\{\left(\frac{r'}{\sqrt{r^2+r'^2}}\right)_4 - \left(\frac{r'}{\sqrt{r^2+r'^2}}\right)_5\right\}\delta r_4$$

$$+ \int_{v_0}^{v_2} M\,\delta r\,dv + \int_{v_3}^{v_4} M\,\delta r\,dv + \int_{v_5}^{v_1} M\,\delta r\,dv, \tag{4}$$

where

$$M = r + \frac{ar}{\sqrt{r^2+r'^2}} - \frac{d}{dv}\frac{ar'}{\sqrt{r^2+r'^2}}. \tag{5}$$

Then to make the terms without the integral sign vanish, we must have $r_2' = r_3'$ and $r_4' = r_5'$, which agrees with the result from equation (2). We also know that the circular arc will, so far as it extends, reduce $M$ to zero, so that the second integral in (4) will vanish, leaving only the rectilinear portions to be examined.

Now along either of these lines $r \cos v$ is constant, so that by differentiation we find

$$r' = \frac{r \sin v}{\cos v} = r \tan v,$$

$$\sqrt{r^2+r'^2} = r\sqrt{1+\tan^2 v} = r \sec v = \frac{r}{\cos v},$$

$$\frac{r}{\sqrt{r^2+r'^2}} = \cos v, \quad \frac{r'}{\sqrt{r^2+r'^2}} = \sin v, \quad \frac{d}{dv}\frac{r'}{\sqrt{r^2+r'^2}} = \cos v.$$

Therefore along either of the rectilinear portions $M$ reduces to $r$. But for these boundaries $\delta r$ is always negative, so that $\delta U$ becomes a negative quantity of the first order.

Hence, if we vary the whole line, we are sure of a maximum without examining the terms of the second order; but if we vary the arc only, such examination would be necessary. In this case we can again employ plane co-ordinates, and we have already shown that $\delta U$ would then become a small negative quantity of the second order.

**239.** If $y_1$ and $y_0$ be not equal, the arc in a continuous solution cannot equal the semi-circumference having as its diameter the line joining the fixed points. Let $A$ and $B$ be the fixed points, and let $y_0$, the ordinate of $A$, be less than $y_1$, the ordinate of $B$. Let $AC$ be drawn parallel to $x$, $C$ being upon the ordinate $y_1$, and bisect $AB$ at $D$ by the perpendicular $DE$, $E$ being on $AC$.

Then the limit of the continuous solution will be reached when the arc becomes tangent to the ordinate $y_0$; that is, when its tangent at $A$ is perpendicular to $AC$. Then it is evident that the centre of the circle will be at $E$. Now $s$ being the length of the arc, and $R$ its radius, we shall have the following equations:

$$AD = \frac{1}{2}\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

$$R = AD \sec EAD = AD\sqrt{1 + \tan^2 EAD},$$

$$\tan EAD = \frac{y_1 - y_0}{x_1 - x_0}.$$

Then $s$ can be determined by equation (10), Art. 91. Denote this particular value of $s$ by $s'$. Then if $l$, the length of the given line, be somewhat greater than $s'$, the line must be first extended along the ordinate $y_0$, produced a certain distance $l'$, until a point is reached at which the same construction can be made as at $A$. Then all the equations just given will be rendered true by merely substituting for $y_0$, $y_0 + l'$, so

that the new values of $R$ and $s$ may be found in terms of $x_0, y_0, x_1, y_1$ and $l'$, and then we have the additional equation $l = l' + s$, so that, $l$ being given, $l'$ can be also determined.

This construction will hold until

$$l = y_1 - y_0 + \frac{\pi}{2}(x_1 - x_0),$$

when the arc will become a semi-circumference. If then $l$ be still further increased, we must retain the same semi-circumference, but also produce $y_1$ as well as $y_0$ a certain distance $l'$. Then we shall have

$$R = \frac{x_1 - x_0}{2}, \qquad s = \frac{\pi}{2}(x_1 - x_0), \qquad l = y_1 - y_0 + 2l' + s.$$

Hence, as $l$ is supposed to be given, $l'$ will be determined, and this construction will hold when $l$ is indefinitely extended.

We must, in closing, call attention to the fact that this problem, when discussed by plane co-ordinates as at the beginning, affords another instance to show that necessary discontinuous solutions are generally suggested by the fundamental equation, even when they do not satisfy at all the equation $M = 0$.

### Problem XLI.

**240.** *It is required to determine the discontinuous solution in Prob. XIX.*

It will be remembered that when $x_0$ is zero, $x_1$ becomes a definite function of the given volume, so that if we require the second point on the axis of $x$ to be fixed—that is, $x_1$ to have a given value—then, unless that value happen to satisfy the equation $x_1 = \sqrt[8]{\frac{15v}{4\pi}}$, where $v$ is the volume, we must resort to some discontinuous solution, if any solution be pos-

sible. (See equation (8), Prob. XIX., observing that $c$ there was shown to equal $x_1$.)

Now as $\delta U$ in this problem does not admit of the usual transformation, because it contains no variation but that of $y$, the fundamental equation is found by equating to zero the coefficient of $\delta y\, dx$ in equation (2) of that problem, which gives either $y = 0$, or else equation (3).

This suggests that if the value of $x_1$ be too great—that is, greater than $\sqrt[8]{\frac{15v}{4\pi}}$ — the solution will consist of a curve satisfying equation (4), and extending from the origin to some point $x_2$ on the axis of $x$, $x_2$ being less than $x_1$, and then of the axis itself from $x_2$ to $x_1$; and that if $x_1$ be too small, the solution may consist of the same solid extended to $x_2$ beyond $x_1$, and then of the axis from $x_2$ to $x_1$, the solutions thus being similar to those in the case of the sphere.

Now the terms of the second order, as we see from equation (2), are

$$\delta U = \int_{x_0}^{x_1} \left\{ a + x\frac{x^2 - 2y^2}{2(x^2 + y^2)^{\frac{3}{2}}} \right\} \delta y^2 dx.$$

But if we put $y = 0$, and for $a$ its value $\frac{-1}{2c^2}$, we shall obtain

$$\delta U = \int \left\{ \frac{-1}{2c^2} + \frac{1}{2x^2} \right\} \delta y^2 dx,$$

where the integral extends over the rectilinear portion only; while if we vary the generating curve, $\delta U$ will take the form given in equation (11), where the integral will extend from $x_0$ to $x_2$, and will be negative whether $x_2$ be less or greater than $x_1$. Hence, observing that $c = x_2$, the entire variation may be written

$$\delta U = \int_{x_0}^{x_2} \frac{3}{2x_2^{\frac{10}{3}}} (x^{\frac{4}{3}} - x_2^{\frac{4}{3}})\, \delta y^2 dx + \int_{x_2}^{x_1} \left\{ \frac{-1}{2x_2^2} + \frac{1}{2x^2} \right\} \delta y^2 dx.$$

Now in order that $U$ may be a maximum, the second integral in (1) must also become negative, otherwise the sign of $\delta U$ would become ambiguous. But any element of this integral will evidently become negative or positive according as $x_2$ is less or greater than $x$. Now when the solid does not extend to the second fixed point, $x$ for the rectilinear part is greater than $x_2$, and the same will be true when the solid extends beyond the second fixed point, provided we agree, as explained in Art. 195, to regard $x$ for the rectilinear part as still increasing from $x_2$ to $x_1$; so that under this supposition we have always a maximum.

**241.** But the solution in the case in which the solid extends beyond the second fixed point may not, perhaps, be deemed altogether satisfactory. For in the volume which is generated by the derived curve, we are obliged, as before, in the case of the sphere, to reckon twice that generated by $\delta y$ along the rectilinear part, and also to regard its attractive force, when counted the second time, as what it would be if each element were placed as far beyond $x_2$ as it now falls short of that point.

We do not, therefore, in reality, compare the attraction exerted by the primitive solid with that which would really be exerted by the derived solid, but merely with what the attraction of that solid would be if the attraction of any particle could vary inversely as the square of the estimated value of $x$, instead of its actual value.

Thus we have here merely a sort of theoretical or imaginary solution, not properly capable of geometrical representation, and presenting itself possibly somewhat as do imaginary roots in the theory of ordinary equations. But the condition that the solid is to meet the axis of $x$ at a second fixed point may, as Prof. Todhunter has suggested, be more naturally understood to mean that the solid is not to stretch beyond the line whose equation is $x = x_1$. Then $c$ in (4) would no longer

be equal to $x_1$, but could be determined from equation (7) by making the limits 0 and $x_1$, since $x_1$ and $v$ are both given; and then all the conditions of the question could be fulfilled.

But should neither of these solutions prove satisfactory, we are still at liberty to suppose that there is no solution, since it is evidently possible to assume such conditions in any problem as will render any solution either continuous or discontinuous impossible; as, for example, if in Prob. XV. we should assume the given line to be shorter than the right line joining the two fixed points.

### Problem XLII.

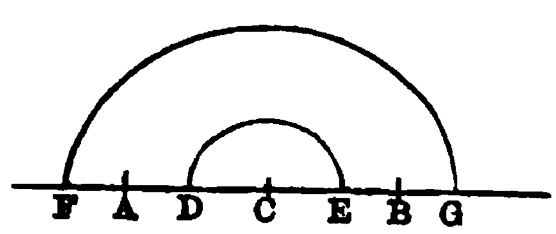
**242.** *It is required to discover the nature of the discontinuous solution in Prob. XXI*

Here, as will appear from reference to the problem, the continuous solution consists of an oblate spheroid whose major axis is to the minor as $\sqrt{2}$ is to 1; that is, whose eccentricity is $\frac{1}{\sqrt{2}}$, $a^2$, the square of the semi-minor axis, being equal to $x_1^2$. Hence, if the given volume be greater or less than $\frac{8\pi x_1^3}{3}$, the solution, if any exist, must be discontinuous.

But the fundamental equation in this case, as will be seen from equation (4), is

$$y(y^2 + 2x^2 - 2a^2) = 0,$$

which gives either $y = 0$ or equation (5), which is the equation of the generating ellipse. Let $A$ and $B$ be the two fixed



points on the axis of $x$, and $C$ the origin, which, it will be remembered, was required to be midway between $A$ and $B$. Then it is suggested that the discontinuous solution might be that represented in the figure, where the generating ellipse is $DE$ or $FG$, according as the given volume is less or greater than $\frac{8\pi x_1^3}{3}$.

Here, then, $U$ for either case may be divided into three integrals, extending respectively from $x_0$ to $x_3$, from $x_3$ to $x_4$, and from $x_4$ to $x_1$; $x_3$ being in the first case the abscissa of $D$, and in the second that of $F$, and $x_4$ being that of $E$ or $G$. We must also recollect that in the second case $x_0$ and $x_1$ are thus estimated:

$$x_0 = -(CF + FA) \qquad \text{and} \quad x_1 = CG + GB.$$

Now we have seen (Art. 120) that the terms of the second order for the ellipse reduce to $\int y^2 \delta y'^2 dx$, and to obtain the variation of the rectilinear portions we have merely to make $y$ zero in the first equation of that article, so that we have

$$\delta U = \int_{x_0}^{x_3} (x^2 - a^2)\delta y'^2 dx + \int_{x_3}^{x_4} y^2 \delta y'^2 dx + \int_{x_4}^{x_1} (x^2 - a^2)\delta y'^2 dx.$$

To render the first and third integrals positive, we must have $x^2 > a^2$; and since $a^2 = x_3^2 = x_4^2$, if we estimate $x$ for the rectilinear part as already explained, this condition will be fulfilled in either case, and $U$ becomes a minimum.

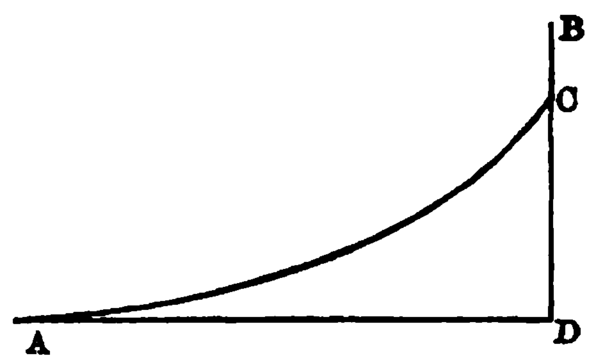
But since the solids generated by both the primitive and the derived curve are to be revolved about the axis of $y$, it must appear that when the solid extends beyond $A$ and $B$ the solution, like that of the preceding problem, is merely theoretical or imaginary. These problems also resemble each other, and differ from all others which we have considered, in that, as $U$ contains $x$ and $y$ only, there are no terms in $\delta U$

without the integral sign; and hence the equation $L=0$ gives, without integration, the equation of the required curve, and there are no terms to consider at the limits.

## Problem XLIII.

**243.** *It is required to determine what discontinuous solutions can present themselves in the discussion of Prob. XX.*

Here the continuous solution is an hypocycloid, in which the radius of the rolling circle is one third that of the fixed circle. But, by the closing remarks of Art. 116, it appears that this solution cannot hold when the given volume becomes less than $\frac{\pi b^3 \sqrt{3}}{5}$, where $b$ is the radius of the given base; so that if the given volume be less than this quantity, the solution, if there be any, must be discontinuous.




Let $AD$ be the axis of $x$, and $DB$ the radius of the generating base. Then, since the volume was to be upon the given base, we would naturally infer that when the volume becomes too small, the generating curve would consist of an arc $AC$ of an hypocycloid, and a portion $CB$ of the radius of the base. In fact, we may understand the conditions of the problem to imply that the solid is always to be upon a portion of the base.

**244.** Before considering whether this solution is also suggested by the calculus of variations, we will show that it is in some cases the solution required.

For the solid generated by $AC$ the resistance will evidently be $2\pi\int_{x_0}^{x_1}\frac{yy'^3}{1+y'^2}dx$, and for the ring generated by $CB$ it will be $\pi(b^2-y_1^2)$. Hence we may minimize the expression

$$U=b^2-y_1^2+2\int_{x_0}^{x_1}\left\{\frac{yy'^3}{1+y'^2}+2ay^2\right\}dx, \tag{1}$$

where we are to regard $y_1$, the ordinate of $C$, as variable, but the other terms at the limits as fixed. Now taking the variation of $U$ under this supposition, transforming it as usual, and making $M$ vanish, we shall obtain, as in Prob. XX., equation (4), which will be of course the differential equation of the hypocycloidal arc $AC$. But we have, after making $M$ vanish,

$$\delta U=-2y_1\delta y_1+2\left\{\frac{3y'^2+y'^4}{(1+y'^2)^2}\right\}_1 y_1\delta y_1=0;$$

and to satisfy this equation, we must, since $y_1$ is not zero, have

$$-1+\frac{3y'^2+y'^4}{(1+y'^2)^2}=0,$$

which gives $y'=\pm 1$. Thus it appears that the generating curve must meet the ordinate of $B$ at an angle of $45^\circ$.

**245.** To determine the sign of the terms of the second order, we must observe that the terms under the integral sign in the value of $U$ given in (1) equal $2U$ in Prob. XX. Hence we shall obtain from the variation of these terms twice the second member of equation (19), Art. 117. But we shall also obtain from this integral a term without the sign of integration. For (19) was obtained under the supposition that $\delta y_1$ and $\delta y_0$ vanish. When, however, this is not the case, we must, as we see from equation (6), Prob. VIII., add to the second member of (19) the terms

$$\frac{1}{2}(f_1'\delta y_1^{\,2}-f_0'\delta y_0^{\,2}). \tag{2}$$

which will give in this case the additional term

$$f_1'\delta y_1^2 \quad \text{or} \quad 2\left\{\frac{3y'^2+y'^4}{(1+y'^2)^2}\right\}_1 \delta y_1^2 \quad \text{or} \quad 2\delta y_1^2;$$

and as this is cancelled by the term of the second order arising from the variation of $-y_1^2$ in $U$, $\delta U$ becomes merely twice the second member of (19), which is positive.

**246.** Thus we have a minimum if $c$ have any real value. Now because $y_1' = \pm 1$, taking the positive sign, we have, from equation (4), Prob. XX., which, it will be remembered, is the fundamental equation in this case also, $y_1 = \frac{c}{4}$; and it is also shown by operations of the differential and integral calculus only, that the given volume, $v'$, will in this case be

$$v' = \frac{13\pi c^3}{1920}. \tag{3}$$

Hence, when $v'$ is given, $c$ and $y_1$ are at once determined.

Now $v'$ can be given as small as we please, but it cannot be as great as we please. For $y_1$ must not exceed $b$; and as $c = 4y_1$, $v'$ evidently increases as we increase $y_1$, and must have its greatest value when $y_1 = b$—that is, when $c = 4b$—and then (3) gives

$$v' = \frac{13\pi b^3}{30}. \tag{4}$$

We see, then, that if the given volume be less than $\frac{\pi b^3 \sqrt{3}}{5}$, we must always employ the discontinuous solution; if it be greater than $\frac{13\pi b^3}{30}$, we must always employ the continuous solution; but if it lie between these values, then we shall have two minima, and must determine which will give the smaller

resistance. This determination must, however, as in former cases, be effected by the ordinary calculus alone, using, of course, any equation which has been thus far obtained.

It will be sufficient here to give the necessary formulæ and results. Let $v_1$ denote the angle whose tangent is $y_1'$. Then, $R$ being the resistance, Prof. Todhunter shows, by methods of the ordinary calculus, that for the continuous solution

$$R = \frac{\pi b^2}{\cos^2 v_1}\left(\frac{3 \sin^2 v_1}{4} - \frac{4 \sin^4 v_1}{5}\right), \tag{5}$$

and that for the discontinuous solution

$$R = \pi\left(b^2 - \frac{13c^2}{30}\right), \tag{6}$$

where, since $v'$ is supposed to be a given quantity, $v_1$ can be determined from equation (12), Art. 116, and $c$ from equation (3) of this article. Now if we take the extreme values of $v'$, for which two solutions are possible,

$$v' = \frac{\pi b^2 \sqrt{3}}{5} \quad \text{and} \quad v' = \frac{13\pi b^2}{30}, \tag{7}$$

we shall find that the two solutions coincide for the first, $R$ being in either case $\frac{7\pi b^2}{20}$, and for the second value of $v'$ we shall find that $R$ is less for the discontinuous than for the continuous solution. For we have in the first case

$$R = \pi b^2\left\{1 - \frac{3\sqrt[4]{52}}{20}\right\} = \pi b^2 \times .44012, \text{ nearly};$$

and in the second

$$R = \frac{9\pi b^2}{20}.$$

It is also shown, by determining the sign of $\frac{dR}{dv'}$, that both for the continuous and discontinuous solution $R$ decreases as $v'$ increases. Hence, from what has been already shown, it will appear that, when there are two solutions, the discontinuous is that which will always give the smaller resistance.

**247.** It will be remembered that in Prob. XX. we considered only the case in which $v$ is supposed to be zero when $y$ is zero. But if we supposed that when $y$ is zero $v$ is $\frac{\pi}{2}$, and measure the arc $s$ from that point, then we shall have, from equation (10) of that problem, $s = -\frac{c}{3}\cos 3v$.

Here, on account of the infinite value of $y'$, our investigation of the terms of the second order will not be satisfactory, and we will therefore adopt $y$ as the independent variable. Then $U$ becomes

$$U = \int_{y_0}^{y_1}\left\{\frac{y}{1+x'^2} + 2ay^2x'\right\}dy. \tag{8}$$

Hence, to the second order,

$$\delta U =$$

$$\int_{y_0}^{y_1}\left\{\frac{-2yx'}{(1+x'^2)^2} + 2ay^2\right\}\delta x'dy + \int_{y_0}^{y_1} y\frac{3x'^2-1}{(1+x'^2)^3}\delta x'^2 dy. \tag{9}$$

Therefore, by making the terms of the first order vanish,

$$2ay^2 - \frac{2yx'}{(1+x'^2)^2} = \text{a constant, which must be } 0;$$

and this must, of course, lead to the hypocycloid, as before.

Then, as the terms of the first order vanish, we shall have

$$\delta U = \int_{y_0}^{y_1} y\frac{3x'^2-1}{(1+x'^2)^3}\delta x'^2\,dy,$$

which is negative so long as $x'^2$ does not exceed $\frac{1}{3}$; that is, so long as $v_1$ is not less than $\frac{\pi}{3}$. Thus in this case the resistance becomes a maximum, provided we can determine real values for $c$.

Now, as before, it is shown that in this case

$$v' = \pi c^3 \cos v_1 \frac{\frac{\cos^8 v_1}{3} - \frac{13 \cos^6 v_1}{10} + \frac{15 \cos^4 v_1}{8} - \frac{7 \cos^2 v_1}{6} + \frac{1}{3}}{\sin^6 v_1}. \tag{10}$$

Also, because equation (5), Prob. XX., holds, we shall find that here, as in equation (13) of the same problem,

$$b = \sin^3 v_1 \cos v_1, \tag{11}$$

and from these two equations $c$ must be determined. It is evident that $v'$ can be made as small as we please; but it cannot be taken as great as we please, because it decreases with $v_1$; and in order to have a maximum, $v_1$ must not be less than $\frac{\pi}{3}$. But when $v_1 = \frac{\pi}{3}$, we shall find

$$v' = \pi c^3 \frac{217\sqrt{3}}{1215}, \tag{12}$$

which is therefore the greatest admissible value of $v'$.

**248.** We are naturally led to inquire whether there will be any discontinuous solution when $v'$ exceeds the value just given.

Since the solid is to be bounded by the given base, the only suggestion which presents itself is that $y_1$ may now be greater than $b$. Then, when $y$ is the independent variable, $U$ will have the form given in (8). But now, as $y_1$ is variable, we must, when we vary $U$, increase also the limit $y_1$ by $dy_1$;

that is, we must add to the terms of the first order in (9) the term

$$V_1\,dy_1 \quad \text{or} \quad \left\{\frac{y}{1+x'^2}+2ay^2x'\right\}_1 dy_1.$$

Now the coefficient of $\delta x_1$ will necessarily vanish, but we cannot also make $V_1$ vanish. Hence $\delta U$ to the first order will not vanish; and as $dy_1$ may have either sign, $U$ will be neither a maximum nor a minimum.

**249.** A somewhat curious point is here noticed by Prof. Todhunter, which it may be useful to consider.

Let $A$ be the distance of the base from the origin. Then we may evidently consider the solid as composed of cylindrical shells whose radius is $y$, thickness $dy$, and length $A-x$. Then, instead of

$$\int_{x_0}^{x_1}\pi y^2 dx \qquad \text{or} \qquad \int_{y_0}^{y_1}\pi y^2 x' dy,$$

the volume may be written $\int_{y_0}^{y_1}2\pi y(A-x)dy$. Therefore with this value of $v'$ we are to maximize or minimize the expression

$$U=\int_{y_0}^{y_1}\left\{\frac{y}{1+x'^2}+aAy-axy\right\}dy. \tag{12}$$

$$\delta U=-\left\{\frac{2yx'}{(1+x'^2)^2}\right\}_1\delta x_1+\left\{\frac{2yx'}{(1+x'^2)^2}\right\}_0\delta x_0$$

$$+\int_{y_0}^{y_1}\left\{-ay+2\frac{d}{dy}\frac{yx'}{(1+x'^2)^2}\right\}\delta x\,dy. \tag{13}$$

Hence, by integration, we obtain

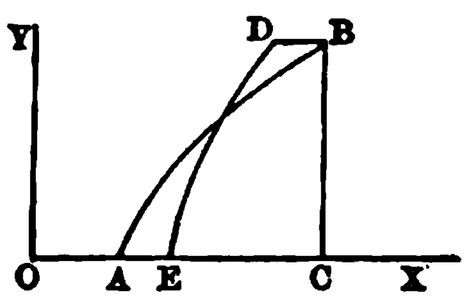
$$-ay^2+\frac{4yx'}{(1+x'^2)^2}=\text{a constant, which must be } 0.$$

This equation is in reality the same as that which we obtained before, and leads, therefore, to the hypocycloid. Thus the integral in (13) will vanish, and so also will the terms at the lower limit, because there $y$ is zero; but the terms at the upper limit will not vanish, so that we have, by the last equation,

$$\delta U = -\frac{1}{2} a y_1^2 \delta x_1.$$

Now since the base is a boundary which we may not pass, $\delta x_1$ is essentially negative, and thus $\delta U$ becomes a positive quantity of the first order, indicating that we have a conditioned minimum, which result would seem to show that we can never have a solid of maximum resistance, thus conflicting with what has been before proved.

**250.** To explain this difficulty, let $AB$ be the primitive curve, and suppose we wished to pass to a derived boundary $EDB$, where $DB$ is parallel to $x$, and infinitesimal.




Then we could not derive this boundary from $AB$ by infinitesimal changes in $y$ and $y'$, although we could by such changes in $x$ and $x'$.

This assertion, which Prof. Todhunter takes no pains to establish, may at first appear incorrect, because we seem to have given $x_1'$ a finite variation in order to obtain $DB$, which would be inadmissible. But the position appears to be sound, since we should regard $x_1'$, after being varied, not as the tangent of the inclination of $DB$ to $y$, but as that of the inclination to $y$ of the tangent to the derived curve at $D$, supposing

the curve $ED$ produced beyond $D$. Then $\delta x_1'$ need not be finite.

Now since $y_1$ is fixed, we shall (unless in the last article we make $\delta x_1$ zero, in which case all the terms of the first order will vanish, and there will be no difficulty) be obliged to pass to a derived curve terminated by $DB$, $DB$ being numerically equal to $\delta x_1$. Still, so long as we adopt for the volume, as we did in (8), the expression $\pi\int_{y_0}^{y_1} y^2 x' dy$, we cannot pass to such a boundary as we have been considering; because although the expression just given will represent the volume generated by the primitive, still, when we change $x'$ into $x' + \delta x'$, and write $v' = \pi\int_{y_0}^{y_1} y^2(x' + \delta x')dy$, $v'$ can only represent the volume generated by $ED$, neglecting entirely that generated by $DB$.

Hence we conclude that the form of $v'$ adopted in (8) is not general enough to permit of a full discussion, as it will not allow every change in the form of the solid which the calculus of variations would in this case sanction. We see, also, that we can have a solid of maximum resistance only under the condition that $y_1$, the radius of the generating base, shall be invariable, and that the curved part of the solid shall always extend to the circumference of the base.

**251.** We have in this discussion a remarkable confirmation of the principle often before stated—that when by variations we have obtained conditions which render any definite integral $U$ a maximum or a minimum, we are not necessarily warranted in asserting more than that $U$ is a maximum or a minimum with respect to admissible variations. For the solid of minimum resistance which we obtained in Prob. XX. is not the solid of least resistance, since by taking a zigzag boundary it could be still further diminished, although we could not pass to such a boundary by the calculus of variations. Moreover, our solid of maximum resistance is such so long only as

we do not make such a change in the form of the solid as in Art. 240. But a solid of still greater resistance would evidently be obtained by passing to a boundary in which $y'$ is alternately zero and infinity, although such a change of form cannot be effected by the calculus of variations.

**252.** It will be remembered that in Art. 243 we were led to the discontinuous solution, which we subsequently verified, by the consideration that the given base constituted a boundary, and that therefore it would probably form some portion of the solution.

Now we have found hitherto the boundaries to be also in some manner suggested by the fundamental equation which is usually the first integral of the equation $M = 0$, even when these boundaries do not in reality cause $M$ to vanish at all. In the present case, however, the discontinuous solution does not appear to be very clearly suggested by the calculus of variations alone, unless, indeed, $U$ can be put under some form different from those which we have yet examined. For, adopting in succession $x$ and $y$ as the independent variable, the first integral of the equation $M = 0$ will be in each case the most general form of the fundamental equation, and we shall have

$$ay^2 - \frac{yy'^3}{(1 + y'^2)^2} = \text{a constant}$$

and

$$ay^2 - \frac{yx'}{(1 + x'^2)^2} = \text{a constant},$$

which constant must, in either case, be zero, because the curve is to meet the axis of $x$. Therefore, rejecting the solution $y = 0$, we have

$$y = \frac{y'^3}{a(1 + y'^2)^2} = \frac{cy'^3}{(1 + y'^2)^2} \quad \text{and} \quad y = \frac{x'}{a(1 + x'^2)^2} = \frac{Cx'}{(1 + x'^2)^2},$$

and these equations lead to the same solution.

Now $y' = \infty$ or $x' = 0$ are not solutions of these equations, unless, indeed, we could suppose $c = \infty$ and $C = \infty$. But these constants will not be infinite for the curve; and since they are in each case the reciprocal of $a$, if we remember that even in a discontinuous solution the constant introduced by Euler's method cannot, like a constant of integration, have two values, it will appear that $c$ and $C$ cannot become infinite at all.

**253.** There would seem to be nothing surprising in the fact that the fundamental equation does not always suggest a boundary which does not cause $M$ to vanish at all, and indeed it would appear more remarkable that such boundaries are so frequently suggested. Cases, however, like the present appear to be rare, and we have now had abundant proof that the calculus of variations does usually suggest solutions when they are possible, and even when such suggestions would not naturally be expected.

Moreover, in discontinuous solutions it very often happens that a trial solution is easily reached without the aid of variations, or at least without examining the form of $M$; and then the calculus of variations affords us the means of verifying or falsifying this proposed solution, and that, too, very frequently without an appeal to the terms of the second order.

**254.** The subject of the present section has been most elaborately treated in the Adams Essay, or Researches in the Calculus of Variations, published by Prof. Todhunter in 1871, and to his labors its present degree of perfection is chiefly due. In this section, which is little more than a condensed view of that treatise, we have endeavored to present all the leading points of that work, and particularly those points which were new to our science. All the examples, therefore, of this section have, with slight modifications, been taken from this essay, although we have in no respect followed its

order. We therefore earnestly recommend the work to all who wish to become fully acquainted with this subject.

We have, with the exception of Prob. XXXIII., considered those cases only in which the discontinuity may be supposed to arise from conditions unconsciously imposed, or at least imposed without seeking to produce it; because it is only when it thus presents itself that its origin can be a source of difficulty. It must, however, be evident that even when no discontinuity would naturally occur in a problem, we can easily impose such restrictions as will render a discontinuous solution necessary, and the work to which we have referred is occupied largely with such examples, some of which exhibit much ingenuity. But as these examples, while affording excellent practice in this department of analysis, present nothing which has not been already considered, it will be sufficient in closing to subjoin one, which is all that our space will permit.

## Problem XLIV.

**255.** *It is required to find the path of quickest descent of a particle from a fixed point $A$ to a second fixed point $B$, under the condition that the path is not to pass without a given circular arc $AB$, which is not to exceed a quadrant; the particle starting from a state of rest at $A$, and $B$ being the lowest point of the arc.*

Assume the horizontal as the axis of $x$. Then, as in Case 2, Prob. II., we shall have

$$\left.\begin{aligned} U &= \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx = \int_{x_0}^{x_1} V dx, \\ \delta U &= P\delta y + \int M \delta y \, dx, \end{aligned}\right\} \qquad (1)$$

where the limits and suffixes are for the present omitted, and

$$\left.\begin{aligned} P &= \frac{dV}{dy'} = \frac{y'}{\sqrt{y(1+y'^2)}}, \\ M &= N - \frac{dP}{dx} = -\frac{\sqrt{1+y'^2}}{2y^{\frac{3}{2}}} - \frac{dP}{dx}, \end{aligned}\right\} \qquad (2)$$

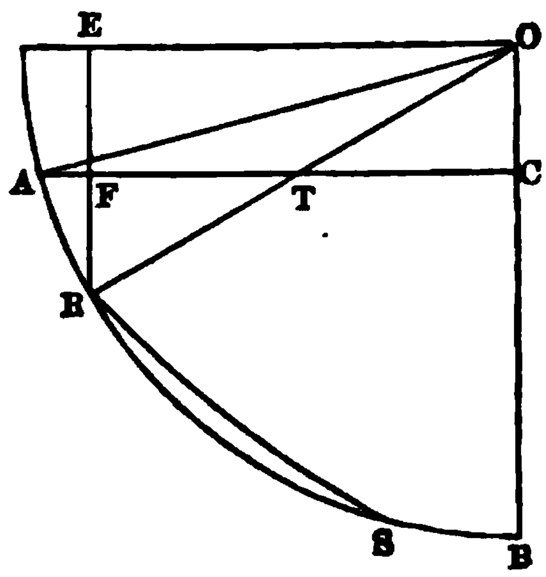
where $N = \frac{dV}{dy}$. Now wherever the sign of $\delta y$ is unrestricted, $M$ must vanish, and this will lead to a cycloid having its cusps on the horizontal line through $A$, and its vertex downward.

But the cycloid alone can never be the solution, because its tangent at $A$ being perpendicular to $x$, it is initially without the circle. Since, then, the circle is the only boundary along which the sign of $\delta y$ can be fettered, the solution must consist either of the given circular arc alone, or of, first, a portion of that arc beginning at $A$, followed by some combination of portions of that arc and cycloidal arcs given by the equation $M = 0$.

**256.** Let the initial and the first cycloidal arc meet at the point $x_2, y_2$. Then there will evidently arise in $\delta U$, as given in (1), the terms $(P_2 - P_1)\,\delta y_2$, and this must either become positive or vanish; that is, since $\delta y_2$ must be negative, $P_2 - P_1$ must be negative or vanish. But if it were negative, we would, as appears from (2), have $y_1' > y_2'$, which would require the cycloid at that point to pass without the circle, which is inadmissible. Hence the coefficient of $\delta y_2$ being zero, we have $y_2' = y_1'$, and the circle and cycloid must be tangent at the point $x_2, y_2$. In like manner they would evidently be tangent if they could meet at any other point.

**257.** Let $AC$ be the horizontal through $A$, $O$ the centre

of the given circular arc, and $r$ its radius, $R$ being the point $x_2, y_2$, so that $RT$ is a normal to the cycloid.




Now take any point on $AC$ as the origin. Then the equation of the circular arc is

$$(x - c)^2 + (y + b)^2 = r^2, \tag{3}$$

where $b = OC$, and $c$ is the abscissa of $O$. Therefore, for the circle, we find

$$y' = -\frac{x - c}{y + b}, \quad \sqrt{1 + y'^2} = \frac{r}{y + b}. \tag{4}$$

Hence, from (2), we have

$$\left.\begin{aligned} N &= \frac{-r}{2y^{\frac{3}{2}}(y + b)} = \frac{-r^2}{2ry^{\frac{3}{2}}(y + b)}, \qquad P = -\frac{x - c}{r\sqrt{y}}, \\ \frac{dP}{dx} &= \frac{-2y + y'(x - c)}{2ry^{\frac{3}{2}}} = \frac{-2y(y + b) - (x - c)^2}{2ry^{\frac{3}{2}}(y + b)}. \end{aligned}\right\} \tag{5}$$

Whence, putting in $N$ the value of $r^2$ from (3), we have

$$M = \frac{y - b}{2ry^{\frac{3}{2}}}. \tag{6}$$

Now this value of $M$ must either vanish or become negative in order that $\int M\delta y\,dx$ may be positive along the circle, since

$M$ will vanish along the cycloid, and this requires merely that $y - b$ shall not become positive.

**258.** Let $t$ and $a$ denote respectively the angles $OTC$ and $OAC$. Then

$$RT = OR - OT = r - \frac{b}{\sin t} = r - \frac{r \sin a}{\sin t}; \qquad (7)$$

and because $RT$ is a normal to the cycloid, we have, $D$ being the diameter of the generating circle,

$$D = \frac{RT}{\sin t} = r\,\frac{\sin t - \sin a}{\sin^2 t}. \qquad (8)$$

If the cycloid can meet the circle again at some other point $S$, we shall obtain a similar expression for $D$, only $t$ will then denote the angle which $OS$ would make with $AC$, and these expressions would be equal. Hence, regarding $t$ as variable, and writing $v = \dfrac{\sin t - \sin a}{\sin^2 t}$, we must be able to effect that $v$ shall twice have an assigned value, or else the circle and the cycloid cannot meet more than once.

Now we find

$$\frac{dv}{dt} = \cos t\,\frac{2 \sin a - \sin t}{\sin^3 t}. \qquad (10)$$

That is, to render $v$ a maximum or a minimum we must have either

$$2 \sin a - \sin t = 0 \quad \text{ or } \quad \cos t = 0. \qquad (11)$$

Since $\sin t$ cannot exceed unity, if $2 \sin a$ be greater than unity, the first equation cannot be satisfied, and $v$ continually increases as $t$ passes from $a$ to $\frac{\pi}{2}$, and therefore cannot twice have the same value; and the same would be true should $2 \sin a$ equal unity.

Neither can we in this case make the cycloidal arc meet the circle at $R$ and also pass through $B$. For

$$CB = OB - OC = r - OC = r - b = r(1 - \sin a),$$

and $D$, as appears from (8), must be less than $CB$ so long as $t$ is less than $\frac{\pi}{2}$; that is, $\sin t$ less than unity; and hence in this case we must use the circle alone.

Now since $y + b = ER = r \sin t$, and $b = OC = r \sin a$, we have $y - b = r(\sin t - 2 \sin a)$, which is in this case negative, thus rendering $\delta U$ positive for the whole circular arc. But $\sin a = \cos AOB$, so that $2 \sin a$ will be unity when $AB$ is an arc of 60°. We conclude, therefore, that unless the given arc exceed 60°, it is itself the path required.

**259.** Let us next consider the case in which the given arc $AC$ exceeds 60°; that is, in which $2 \sin a$ is less than unity.

Here the first of eqs. (11) is satisfied when $\sin t = 2 \sin a$; and as $v$ then becomes a maximum, it may evidently have the same value twice. But now the value of $y - b$ just given would become positive before we reach the point $B$, and so a part of $\delta U$ would become negative if we suppose the path to terminate with a portion of the circular arc through $B$, which is inadmissible.

We conclude then, in this case, that the required path must consist of the circular arc $AR$ and the cycloidal arc $RB$ tangent to the circle internally at $R$. Then $y - b$ will be negative for the whole circular arc $AR$. For as the cycloid is tangent to the circle internally at $R$, its radius of curvature must at that point be less than $r$; that is, since the radius is twice the normal, $2RT < r$, so that $OT > RT$. Whence $OT \sin TOE = EF = b$ is greater than $RT \sin RTF$ or $RF \sin TOE$; that is, $b > y$ and $y - b$ is negative.

**260.** We must now show that a cycloid can be drawn tangent to the circle internally at $R$ and passing through $B$.

First, assume $D = CB = r - b$, putting the vertex at $B$. Then, since $2(r - b)$, the radius of curvature at $B$, is greater than $r$, the cycloid will be tangent to the circle externally at $B$. But by taking $D$ sufficiently large, the cycloid still passing through $B$, we can cause the cycloid to fall entirely within the circle, and then by diminishing $D$, while retaining $B$ as a cycloidal point, we must arrive at a value of $D$ which will cause the cycloid to become tangent to the circle before cutting it, and this point of contact will be neither at $A$ nor $B$. For at $A$, $y'$ for the cycloid is infinite, while for the circle it is not; and at $B$, $y'$ for the circle is zero, while for the cycloid it is not.

Now as the solution is real, it is unnecessary to discuss the value of $D$ or the position of the point of contact $R$, or of the cusps on $AC$.

**261.** No natural discontinuity presents itself in the discussion of Prob. II., since, if the two fixed points be not in the vertical nor in the horizontal line, we can cause a cycloid to pass through them both, and have its cusps on the horizontal line through the upper point. Neither can there be admissible but unnecessary discontinuity of the kind discussed in Prob. XXXIX. For if there could be two cycloidal arcs meeting at any point, they must, as we have seen, both have their cusps on the horizontal through the point from which the particle starts, and must also, as appears from Art. 256, be tangent. Moreover, from Art. 25, the fundamental equation is $y(1 + y'^2) = a = D$; and since $y'$ has but one value at the point of contact, $D$ can have but one value there for the two cycloids, and the cycloidal arcs must therefore be generated by the same circle rolling on the same horizontal. Now as $y'$ in any cycloid can have a given value but once, these arcs have also their cusps in common; that is, there are not two cycloidal arcs at all.

## SECTION X.

### *OTHER METHODS OF VARIATIONS.*

**262.** Hitherto, whether employing plane or polar coordinates, we have ascribed variations to the dependent variable only and its differential coefficients, adding also, when a change in the independent variable is necessary, an increment to its limiting values only. This method, which has been adopted by the two most elaborate English writers, Profs. Jellett and Todhunter, as also by the chief German writer, Strauch, is undoubtedly the best. But many writers vary the independent variable also throughout the whole definite integral; and as the reader will be likely to meet with this method, the present work would be incomplete if it did not explain this method sufficiently to enable him to follow the solution of any problem in which it may be employed.

*First Method.*

**263.** Suppose we assume the equation

$$U = \int_{x_0}^{x_1} V dx, \tag{1}$$

where $V$ is any function of $x$, $y$, $y'$, etc., and suppose $y$ to become the ordinate of some primitive curve. Then, by varying $U$ in the most general manner, we can pass to any curve which can be derived from the first by infinitesimal changes in $x_0$, $x_1$, $y$, $y'$, etc.

But we may also pass to the same derived curve by moving, without change of value, any ordinate of the primitive curve an infinitesimal distance $\delta x$ along the axis of $x$, and then varying it so as to make it become the ordinate of the derived curve for the new abscissa $x + \delta x$. In this method $\delta y$, $\delta y'$, etc., will mean the difference between $y$, $y'$, etc., for the primi-

tive curve, and corresponding to the abscissa $x$, and the same quantities for the derived curve corresponding to the abscissa $x + \delta x$. Of course for any given value of $x$ we may suppose $\delta x$ to have either sign, or to vanish; and it is evident that when the limits are to be fixed, the latter supposition must be made regarding the quantities $\delta x_0$ and $\delta x_1$.

**264.** We are led, then, to inquire what will be the expression for $\delta U$, when $x$ also is regarded as capable of variation throughout the definite integral $U$.

In (1) change $x$ into $x + \delta x$, $y$ into $y + \delta y$, etc., and let

$$U' = U + \delta U \qquad \text{and} \qquad V' = V + \delta V \tag{2}$$

be the new values of $U$ and $V$. Then, observing that $dx$ will become, by being varied,

$$\delta dx = \frac{d}{dx}(x + \delta x)\,dx, \tag{3}$$

we shall have

$$U' = \int_{x_0}^{x_1} V' \frac{d}{dx}(x + \delta x)\,dx. \tag{4}$$

Whence

$$U' - U = \delta U = \int_{x_0}^{x_1} V' \frac{d}{dx}(x + \delta x)\,dx - \int_{x_0}^{x_1} V dx$$

$$= \int_{x_0}^{x_1} (V + \delta V)\frac{d}{dx}(x + \delta x)\,dx - \int_{x_0}^{x_1} V dx. \tag{5}$$

This is exact; but approximating to the first order only, we have

$$\delta U = \int_{x_0}^{x_1} \frac{V d\delta x}{dx}\,dx + \int_{x_0}^{x_1} \delta V dx. \tag{6}$$

But

$$\int_{x_0}^{x_1} \frac{V d\delta x}{dx}\,dx = V_1\delta x_1 - V_0\delta x_0 - \int_{x_0}^{x_1}\left[\frac{dV}{dx}\right]\delta x\,dx, \tag{7}$$

where brackets denote the complete differential coefficient of $V$; that is,

$$\left[\frac{dV}{dx}\right]=\frac{dV}{dx}+\frac{dV}{dy}y'+\frac{dV}{dy'}y''+\text{etc.}=M+Ny'+Py''+\text{etc.} \quad (8)$$

Moreover, it is evident that, to the first order,

$$\delta V = M\delta x + N\delta y + P\delta y' + \text{etc.} \quad (9)$$

Hence (6) becomes

$$\delta U = V_1\delta x_1 - V_0\delta x_0 + \int_{x_0}^{x_1}\Big\{ N\delta y + P\delta y' + Q\delta y'' + \text{etc.}$$
$$- (Ny' + Py'' + Qy''' + \text{etc.})\,\delta x \Big\}\, dx. \quad (10)$$

**265.** But the formulæ hitherto employed for $\delta y'$, $\delta y''$, etc., will not now hold true, so that we must, before we can further transform (10), ascertain what will be the values of these quantities under the present supposition.

First, in $y'$ change $x$ into $x+\delta x$ and $y$ into $y+\delta y$, and we have

$$\delta y' = \frac{d(y+\delta y)}{d(x+\delta x)} - y' = \frac{y' + \frac{d\delta y}{dx}}{1+\frac{d\delta x}{dx}} - y'$$

$$= y' + \frac{\frac{d\delta y}{dx} - \frac{y'd\delta x}{dx}}{1+\frac{d\delta x}{dx}} - y' = \left(\frac{d\delta y}{dx} - \frac{y'd\delta x}{dx}\right)\left(1+\frac{d\delta x}{dx}\right)^{-1}, \quad (11)$$

which is exact; and to approximate to any order required we have only to develop sufficiently the second factor. Thus, to the second order,

$$\delta y' = \left(\frac{d\delta y}{dx} - \frac{y'd\delta x}{dx}\right)\left(1 - \frac{d\delta x}{dx}\right) \quad (12)$$

or, omitting the terms of the second order,

$$\delta y' = \frac{d\delta y}{dx} - \frac{y'd\delta x}{dx} = \frac{d}{dx}(\delta y - y'\delta x) + y''\delta x. \qquad (13)$$

To obtain to the first order the value of $\delta y''$ we have only to substitute in (13) $y'$ for $y$, $y''$ for $y'$, and $y'''$ for $y''$, so that

$$\delta y'' = \frac{d}{dx}(\delta y' - y''\delta x) + y'''\delta x = \frac{d^2}{dx^2}(\delta y - y'\delta x) + y'''\delta x. \ (14)$$

The Greek letter $\omega$ (omega, or $o$) is usually put for $\delta y - y'\delta x$. Then we shall find

$$\left.\begin{array}{c} \delta y' = \dfrac{d\omega}{dx} + y''\delta x, \qquad \delta y'' = \dfrac{d^2\omega}{dx^2} + y'''\delta x, \\ \delta y^{(n)} = \dfrac{d^n\omega}{dx^n} + y^{(n+1)}\delta x, \end{array}\right\} \qquad (15)$$

which equations are, of course, true to the first order only.

**266.** Now substituting in (10) the values of $\delta y'$, $\delta y''$, etc., derived from (15), that equation will become

$$\delta U = V_1\delta x_1 - V_0\delta x_0 + \int_{x_0}^{x_1}(N\omega + P\omega' + Q\omega'' + \text{etc.})dx, \qquad (16)$$

where $\omega' = \dfrac{d\omega}{dx}$, etc. Here $\omega$, $\omega'$, $\omega''$, etc., take the place of $\delta y$, $\delta y'$, $\delta y''$, etc., in the former method, so that integrating by parts, as in that method, we shall obtain

$$\delta U = V_1\delta x_1 - V_0\delta x_0 + h_1\omega_1 - h_0\omega_0 + i_1\omega_1' - i_0\omega_0' + \text{etc.}$$
$$+ \int_{x_0}^{x_1}(N - P' + Q'' - \text{etc.})\omega\, dx, \qquad (17)$$

where the coefficients of $\omega_1$, $\omega_0$, $\omega_1'$, etc., are the same as those of $\delta y_1$, $\delta y_0$, $\delta y_1'$, etc., in equation (5), Art. 36, $h$, $i$, etc., being

used as in equation (7), Art. 37; while the coefficients of $\omega\, dx$ and $\delta y\, dx$ are also identical.

Moreover, since $dx_0$, $dx_1$ and $\delta x_0$, $\delta x_1$ mean the same thing in the two methods, it appears that $\delta U$ in this case is the same in form as the most general variation of $U$ found by the other method, $\omega$ taking the place of $\delta y$.

**267.** Suppose, now, we wish to discover by this method the conditions which will maximize or minimize $U$. Then it will appear, by the same reasoning as before, that $\delta U$ to the first order must vanish, while the terms of the second order must preserve an invariable sign, becoming negative for a maximum and positive for a minimum. Hence (17) may be written

$$\delta U = L_1 - L_0 + \int_{x_0}^{x_1} M\omega\, dx$$

$$= L_1 - L_0 + \int_{x_0}^{x_1} M\,\delta y\, dx - \int_{x_0}^{x_1} My'\delta x\, dx = 0. \quad (18)$$

Therefore the coefficients of $\delta y\, dx$ and $\delta x\, dx$ are so related that if one vanish the other must vanish also, unless, indeed, $y'$ should become zero throughout the curve.

Now $\delta x$ and $\delta y$ under the integral sign are entirely independent of any conditions which those quantities may be required to fulfil at the limits, and therefore we must have

$$L_1 - L_0 = 0 \quad \text{and} \quad \int_{x_0}^{x_1} M\omega\, dx = 0. \quad (19)$$

But $\omega$, like $\delta y$, is wholly in our power, while $M$ is not, so that we must necessarily, as before, suppose $M$ to vanish, and we can obtain no additional equation by considering separately the integrals in the last member of (18).

**268.** Let us now briefly consider the terms at the limits. Suppose, in the first place, $x_1, x_0, y_1, y_0, \ldots . y_0^{(n-1)}$ to be

fixed; that is, to have no variation. Then $\omega_1$, $\omega_0$, $\omega_1'$, etc., and $\delta x_1$ and $\delta x_0$, will severally vanish. For let $y^{(m)}$ be any differential coefficient of $y$ not higher than $y^{(n-1)}$. Then we have at either limit

$$\delta y^{(m)} = \frac{d^m\omega}{dx^m} + y^{(m+1)}\,\delta x = 0;$$

and $\delta x$ being zero at either limit, we have for that limit

$$\frac{d^m\omega}{dx^m} = \omega^{(m)} = 0.$$

Hence, in this case, $L_1 - L_0$ will vanish, and we must determine the $2n$ constants as we did formerly when all the limiting values were fixed.

Let us next suppose $x_1$ and $x_0$ only to be fixed. Then, at either limit, $\omega = \delta y$, $\omega' = \delta y'$, $\omega'' = \delta y''$, etc., and assuming these quantities to be unrestricted, $h_1$, $h_0$, $i_1$, $i_0$, etc., must severally vanish, which are the same conditions for the determination of the constants as we would have under the same supposition by employing the other method. Neither can we obtain any additional equations by putting for $\omega$, $\omega'$, etc., their values, and then making $\delta x$ at the limits vanish. If we make the limiting values of $y$ also invariable, $\omega_1$ and $\omega_0$ will vanish, all the other conditions remaining as before, so that we shall only lose the equations $h_1 = 0$ and $h_0 = 0$, which will be replaced by the conditions that $y_1$ and $y_0$ must have given values.

Proceeding similarly, it will appear that when $x_1$ and $x_0$ are fixed, the same equations for the determination of the $2n$ arbitrary constants arising from the integration of the equation $M = 0$ will be obtained as would, under the same supposition, have been found by the other method.

Let us, in the last place, suppose that $x_1$ and $x_0$ are also variable. Then, if no restriction be imposed upon any of the variations, we shall have, besides the equations already obtained, $V_1 = 0$ and $V_0 = 0$, and we shall find that we cannot

obtain any other equations. Here the conditions are the same as those noticed in Art. 77, and the $2n+2$ equations cannot in general be satisfied.

But suppose that, as in Prob. IX., the extremities of the required curve are to be confined to two fixed curves whose equations are, as in Art. 69, $y = f$ and $y = F$, $f$ and $F$ being functions of $x$. Here $\delta y$ has not the same meaning as in the former method, so that equations (10), Art. 69, or rather equations (2), Art. 76, will not now be applicable. But it is evident that now $\delta y_1 = f_1'\delta x_1$ and $\delta y_0 = F_0'\delta x_0$; so that we shall now have at the upper limit

$$\omega_1 = (f_1' - y_1')\,\delta x_1, \qquad \omega_1' = \left\{ \frac{d}{dx}(f' - y') \right\}_1 \delta x_1, \qquad (20)$$

and similar equations in $F$ hold for the lower limit. Now observing that $\delta x_1$ and $\delta x_0$ here mean the same thing as $dx_1$ and $dx_0$ in the other method when used to change the limiting values of $x$, we see from equations (2), Art. 76, that for either limit we must substitute the same thing for $\delta y$ in the first method as for $\omega$ in the second, and the same thing for $\delta y'$ in the first as for $\omega'$ in the second; so that the coefficients involved must evidently be the same in both methods. Hence we must always obtain by either method precisely the same equations of condition at the limits.

**269.** Thus it will be seen that the results obtained by the two methods are the same, whether as regards the general solution, or the conditions which must hold at the limits, and that nothing is gained by the latter method, while the labor of obtaining the required results is somewhat increased. This disadvantage will become still more obvious when we seek to examine the sign of the terms of the second order. We shall not, however, enter upon this examination further than to observe that we must, in finding these terms, be careful not to reject any of the terms of the second order. Thus,

after having approximated to the second order in equation (5), if we employ (13) and (14) in transforming the terms of the first order, we must remember that the value of $\delta y'$ which we now require is given by (12), and that (13) and (14) are not sufficiently accurate, and that we must therefore add to the terms already assigned to the second order those which are neglected in the first by the use of (13) and (14); and it is easy to see that this will generally involve us in much difficulty.

It is believed that the foregoing account of the present method will be found sufficient to enable the reader to follow any solution which may be presented, which is all that is necessary, since its adoption, as a mode of original investigation, cannot be advised.

*Second Method.*

**270.** The method which we next proceed to explain possesses oftentimes decided advantages, particularly when we come to consider problems involving three co-ordinate axes, and is moreover that which is adopted by Prof. Jellett in the discussion of geometrical problems. As we shall be obliged to consider it at some length, the reader will, we think, most easily comprehend its nature and use by the consideration of an example.

### Problem XLV.

*It is required to discover the conditions which will maximize or minimize the expression* $U = \int_{x_0}^{x_1} v\sqrt{1+y'^2}\,dx$, *where* $v$ *is any function of* $x$ *and* $y$ *only, and constants, the limits being fixed or variable.*

Now assuming $s$ as the arc of the required primitive curve, $U$ may be written

$$U = \int_{s_0}^{s_1} v\,ds. \tag{1}$$

Let $ab$ be the required arc, and on it take at pleasure any points $c$, $d$, $e$, etc., and regard these points as knots or spots upon a flexible cord. Then, when we make any infinitesimal alteration in the form of $ab$, the arcs $ac$, $ad$, $ae$, etc., will undergo no change in length, but the co-ordinates of the points $c$, $d$, $e$, etc., will in general undergo an infinitesimal change.

But the arcs $ac$, $ad$, etc., are any values of $s$, measured from $a$, so that it appears that we can pass from $ab$ to any derived curve by varying $x$ and $y$ in (1), while regarding $s$, and therefore $ds$, as undergoing no variation.

**271.** Taking the variation of (1) under this supposition, we have

$$\delta U = \int_{s_0}^{s_1} \left\{ \frac{dv}{dx}\,\delta x + \frac{dv}{dy}\,\delta y \right\} ds. \tag{2}$$

But (2) gives the variation of $U$ only under the supposition that we need not make any change in the length of the primitive curve, which is not usually the case. For suppose the required curve be conditioned to always connect two fixed points or two fixed curves. Then if we vary $ab$ without producing any change in its length, we shall in reality reduce the problem to one of relative maxima and minima, in which the length of $s$ is to be fixed, and in which, as we have already shown, the form of the derived curve cannot be wholly unrestricted. If, then, the problem be, as we have assumed, one of absolute maxima and minima—that is, if we are required to vary the form of $ab$ in the most general manner consistent with the method of variations—the arc of the derived curve connecting the given points or given curves will not necessarily have the same length as $ab$. Still it is not necessary to vary $s$ or $ds$ under the integral sign, because we can evidently pass from $ab$ to any derived curve $AB$ by first, before varying $ab$, giving to it increments or decrements at $a$ and $b$ so as to obtain a new arc equal in length to $AB$, and then varying the form of this new arc in the most general manner.

But as these increments must be infinitesimal, we may denote them by $ds_0$ and $ds_1$. Now if in (1) we change the limits into $s_0 + ds_0$ and $s_1 + ds_1$, we may find approximately the change which will result to $U$ in precisely the same manner as if the expression were $U = \int_{x_0}^{x_1} V dx$, and $x_1$ and $x_0$ only were to be varied. Hence this change will be

$$v_1 ds_1 - v_0 ds_0 + \frac{1}{2}\left[\frac{dv}{ds}\right]_1 ds_1^2 - \frac{1}{2}\left[\frac{dv}{ds}\right]_0 ds_0^2 + \text{etc.}, \qquad (3)$$

where brackets denote the total differential coefficients. But we wish to find $\delta U$ to the first order only, so that we may write, as the new value of $U$,

$$U' = U + v_1 ds_1 - v_0 ds_0 = v_1 ds_1 - v_0 ds_0 + \int_{s_0}^{s_1} v ds. \qquad (4)$$

If now we vary the form of the arc in the most general manner, and suppose $U'$ to become $U''$, $U''$ will exceed $U'$ by the second member of (2) increased by $\delta v_1 ds_1 - \delta v_0 ds_0$. Hence, observing that the last two terms, being of the second order, must be rejected, we shall find

$$U'' - U = \delta U = v_1 ds_1 - v_0 ds_0 + \int_{s_0}^{s_1} \left\{ \frac{dv}{dx}\delta x + \frac{dv}{dy}\delta y \right\} ds, \qquad (5)$$

which is the form of $\delta U$ which we must in general employ whether the curve be required to connect two fixed points or two fixed curves.

**272.** As $\delta x$ and $\delta y$ now denote the changes which the coordinates of any point when regarded as fixed on the arc, like a knot on a cord, would undergo, owing to any infinitesimal alteration in the form of the arc, it will, we think, appear after a little reflection that we cannot regard $\delta x$ and $\delta y$ as entirely independent, although we cannot state explicitly the nature of the relation subsisting between them. We can, however,

easily assure ourselves that they are not independent. For if they were, then, since $\delta U$ must vanish in order that $U$ may be a maximum or a minimum, we would have $\frac{dv}{dx} = 0$ and $\frac{dv}{dy} = 0$. Whence

$$\frac{dv}{dx} + \frac{dv}{dy}\frac{dy}{dx} = \left[\frac{dv}{dx}\right] = 0.$$

Therefore we would have as a condition necessary to a maximum or a minimum $v =$ a constant, which is false, since in Prob. VII. we have

$$U = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx = \int_{s_0}^{s_1} y\, ds,$$

and $y$ is not constant.

If we could express $\delta y$ explicitly in terms of $\delta x$ and other quantities, we might eliminate one of the variations, and then equate the coefficient of the remaining variation to zero. But as this cannot be done in the present case, an ingenious method of overcoming this difficulty has been devised by Lagrange, which we now proceed to apply, reserving a general explanation of this method until the reader has become somewhat familiar with its spirit.

**273.** We have always, whether along the primitive curve $ab$ or the derived curve $AB$, $ds^2 = dx^2 + dy^2$, so that

$$x'^2 + y'^2 - 1 = 0, \tag{6}$$

where accents will denote differentiations with respect to $s$; and as this equation must always hold, it follows that the variation of its first member—that is, the change which that member will experience when we change $x$ into $x + \delta x$ and $y$ into $y + \delta y$, $s$ remaining unaltered—will be zero. Hence we must have

$$x'\delta x' + y'\delta y' = 0. \tag{7}$$

Then, when we change $x$ into $x + \delta x$, the new value of $x'$ is

$$\frac{d}{ds}(x + \delta x) \quad \text{or} \quad x' + \frac{d\delta x}{ds}.$$

Hence

$$\delta x' = \frac{d\delta x}{ds}. \tag{8}$$

Similarly, when we change $x'$ into $x' + \delta x'$, the new value of $x''$ is

$$\frac{d}{ds}(x' + \delta x') \quad \text{or} \quad x'' + \frac{d\delta x'}{ds}.$$

Whence

$$\delta x'' = \frac{d\delta x'}{ds} = \frac{d^2\delta x}{ds^2}. \tag{9}$$

In the same way we shall find

$$\delta y' = \frac{d\delta y}{ds}, \qquad \delta y'' = \frac{d^2\delta y}{ds^2}, \qquad \text{etc.} \tag{10}$$

Now these formulæ are analogous to those in Art. 9, and are, unlike those obtained in the preceding method, exact. Moreover, it is evident that equations similar to those just obtained must hold when we have any number of variables $x, y, z, u$, etc., all dependent upon the same independent variable, which is itself incapable of receiving any change from being varied, the limiting values only being supposed to be susceptible of an increment.

**274.** Now because (7), (8) and (10) are true, we may write

$$\int_{s_0}^{s_1} \left\{ lx' \frac{d\delta x}{ds} + ly' \frac{d\delta y}{ds} \right\} ds = 0, \tag{11}$$

where $l$ may be any finite quantity we please, either constant or variable. Transforming (11) in the usual way, and adding the resulting equation to the value of $\delta U$ in (5), we have

$$\delta U = v_1 ds_1 - v_0 ds_0 + l_1(x'\delta x + y'\delta y)_1 - l_0(x'\delta x + y'\delta y)_0$$

$$+\int_{s_0}^{s_1} \left\{ (v_x - \frac{d}{ds} lx')\delta x + (v_y - \frac{d}{ds} ly')\delta y \right\} ds = 0$$

$$= L_1 - L_0 + \int_{s_0}^{s_1} \{M\delta x + N\delta y\} ds, \tag{12}$$

where the suffixes $x$ and $y$ denote partial differentiation with respect to those quantities.

As we have now introduced into $\delta U$ the only connecting equation between $\delta x$ and $\delta y$, and have reduced the result as far as possible, it will appear, by reasoning precisely like that hitherto employed, that since an unrestricted integral cannot equal a given function of quantities relating to the limits only, we must have

$$L_1 - L_0 = 0, \qquad \int_{s_0}^{s_1} \{M\delta x + N\delta y\} ds = 0. \tag{13}$$

Now since $l$ is in our power, suppose it to be so taken as to cause $M$ to vanish throughout $U$. Then the second of equations (13) will become $\int_{s_0}^{s_1} N\delta y\, ds = 0$; and as $\delta y$ is evidently entirely independent of $N$, this equation can only be satisfied by making $N$ also vanish throughout $U$.

**275.** We have then the equations

$$\left. \begin{aligned} v_x - (lx')' = 0 = v_x - lx'' - l'x', \\ v_y - (ly')' = 0 = v_y - ly'' - l'y'. \end{aligned} \right\} \tag{14}$$

Multiply the last members by $x'$ and $y'$ respectively, and add. Then, observing that

$$v_x x' + v_y y' = v', \qquad x'x'' + y'y'' = 0, \qquad x'^2 + y'^2 = 1, \quad (15)$$

we shall obtain

$$v' - l' = 0 \quad \text{and} \quad l = v + c. \quad (16)$$

**276.** Before proceeding further we must fully determine $l$ by ascertaining the value of $c$, which can be done by means of the terms at the limits, which we will next examine.

To prepare our way, we observe, first, that it is immaterial, in passing from a primitive to a derived curve, whether we first increase the limits by the positive or negative increments $ds_0$ and $ds_1$, and then vary the new arc, or first vary the form of the entire arc, and then add these same increments to its extremities.

In the second place, the increments which would result to $x$ or $y$ in passing along any infinitesimal arc, $ds_0$ or $ds_1$, while it belongs to the primitive curve, and also after it has undergone an infinitesimal change of form, but none in length, and has assumed its new position as a part of the derived curve, cannot differ by any term of the first order, although they may differ by a term of the second order.

**277.** Suppose, first, that the curve is to connect two fixed points $A$ and $B$, the required primitive curve being $ab$, so that before it is varied $a$ will be at $A$, and $b$ at $B$, and consider the upper limit. At $b$ add a positive or negative increment $ds_1$, and denote the new arc by $ac$. Also let $x_1 + dx_1$ and $y_1 + dy_1$ be the co-ordinates of $c$; then it is evident that the difference between the co-ordinates of $b$ and $c$—that is, the increments which would result to $x$ and $y$ by passing along the arc from $c$ to $b$—must be $-dx_1$ and $-dy_1$ or $-x_1'ds_1$ and $-y_1'ds_1$. Now vary the form of $ab$. Then the point $b$ will assume a new position whose co-ordinates will be $x_1 + \delta x_1$ and

$y_1 + \delta y_1$, while the point $c$ will now fall upon $B$. Hence $\delta x_1$ and $\delta y_1$ are the increments which $x$ and $y$ receive as we pass from $B$ to $b$ on the derived curve. Therefore, as the arc $Bb$ on the derived curve was the arc $bc$ or $Bc$ on the primitive curve, having, without change of length, merely altered its position and form infinitesimally, it appears, by the second remark of the last article, that, to the first order, we must have

$$\delta x_1 = -x_1' ds_1, \qquad \delta y_1 = -y_1' ds_1; \tag{17}$$

and similar equations would, of course, hold at the lower limit.

**278.** Next, suppose the required curve is to connect two fixed curves whose differential equations are $dy = f'dx$ and $dy = F'dx$, and consider the upper limit.

Let the required primitive curve cut the fixed curves at $b$ before, and at $B$ after it has been varied. Then we can easily find the co-ordinates of $B$ from the first remark of Art. 276. For when we vary the primitive curve, the co-ordinates of the extremity in question will become $x_1 + \delta x_1$ and $y_1 + \delta y_1$; and if now to this extremity we add the positive or negative increment $ds_1$, denoting by $dx_1$ and $dy_1$ the corresponding increments of $x$ and $y$, we shall reach $B$, whose co-ordinates must therefore be $x_1 + \delta x_1 + dx_1$ and $y_1 + \delta y_1 + dy_1$. Subtracting $x_1$ and $y_1$, we find the changes which $x$ and $y$ experience as we pass from $b$ to $B$ along the fixed curve to be respectively $\delta x_1 + dx_1$ or $\delta x_1 + x_1' ds_1$ and $\delta y_1 + dy_1$ or $\delta y_1 + y_1' ds_1$.

But the increment which results to $y$ in passing along the arc $bB$ must be $f'$ times that which results to $x$; so that we must have

$$\delta y_1 + y_1' ds = f_1'(\delta x_1 + x_1' ds_1); \tag{18}$$

and a similar equation in $F'$ can evidently be obtained for the lower limit. Of course these equations, like (17), are true to the first order only, because we have estimated $ds_1$ along the

derived curve, whereas it should be taken along the primitive curve.

**279.** Let us now consider the terms at the limits in (12), first supposing the required curve is to connect two fixed points. Here substituting for $\delta x_1$ and $\delta y_1$, and also for $\delta x_0$ and $\delta y_0$, their values from (17), and observing equation (6), we shall obtain

$$(v-l)_1 ds_1 - (v-l)_0 ds_0 = 0. \tag{19}$$

But $ds_1$ and $ds_0$ are entirely independent, so that their coefficients must severally vanish. Hence we have $l_1 = v_1$, and $c$ in (16) must become zero, giving us $l = v$ throughout the integral.

Next suppose the curve is to connect two fixed curves as in the last article, and consider the upper limit. Substituting the value of $\delta y_1$ found from (18), this limit gives

$$v_1 ds_1 + l_1 x_1'\delta x_1 + l_1 y_1'(f'\delta x + f'x'ds - y'ds)_1; \tag{20}$$

and a similar equation will hold at the lower limit. Now these two limiting equations must be absolutely independent, because we may suppose one extremity of the required curve to be absolutely fixed. We must, therefore, equate (20) to zero.

Now it will appear upon a little reflection that $\delta x_1$ and $ds_1$ must be also entirely independent, so that we may equate their coefficients severally to zero. Hence, if $l_1$ be not zero, (20) will give

$$v_1 + l_1 y_1'(f'x' - y')_1 = 0, \qquad x_1' + f_1' y_1' = 0. \tag{21}$$

Substituting in the first of these equations for $f_1'y_1'$ its value, $-x_1'$, found from the second, and observing equation (6), we obtain, as before, $v_1 - l_1 = 0$; so that here also, as appears from (16), $v = l$.

If $l_1$ should become zero, then, since neither $x_1'$ nor $y_1'$ can become infinite, the upper limiting terms would reduce to

$v_1 ds_1 = 0$, so that $v_1$ must also vanish. Hence, here also, $c$ in (16) vanishes, and we therefore have always $v = l$.

**280.** The reader of Prof. Jellett's work will observe that in Chapter IV., in which he adopts this method, he has, in giving the terms at the limits, uniformly omitted the terms $V_1 ds_1 - V_0 ds_0$, and this omission has led him into an unsatisfactory method of determining the constant $c$, which is in his book $a$, and which, as we have seen, can be determined regularly by the equations at the limits. (See Todhunter's History of Variations, Art. 348.) It happens, however, that his results in discussing by this method the conditions which must hold at the limits are in every case correct, although the method by which they are obtained is certainly not strictly so. The reader will find it profitable to verify this latter assertion, which is made upon the authority of the author alone.

**281.** Let us now return to the general solution. Putting $v$ for $l$ in the last members of (14), we have

$$v_x - x'v' = vx'', \qquad v_y - y'v' = vy''. \tag{22}$$

Now in these equations multiply $v_x$ and $v_y$ by $x'^2 + y'^2$, which is unity, and put in each for $v'$ its value from the first of equations (15). Then, reducing and factoring, we shall obtain

$$y'(v_x y' - v_y x') = vx'', \qquad x'(v_y x' - v_x y') = vy''. \tag{23}$$

Multiplying the first of these equations by $y'$, the second by $x'$, and subtracting the second from the first, remembering equation (6), we have

$$v_x y' - v_y x' = v(y'x'' - x'y''). \tag{24}$$

Let $r$ be the radius of curvature. Then we know that $y'x'' - x'y'' = \frac{1}{r}$. Hence we may write

$$\frac{1}{r} = \frac{1}{v}(v_x y' - v_y x') = -\frac{1}{v}(v_x \cos A + v_y \cos B), \qquad (25)$$

where $A$ is the angle which the normal makes with the axis of $x$, and $B$ the acute angle which it makes with the axis of $y$.

It is impossible to proceed further with the solution so long as the form of $v$ is wholly undetermined; but equation (25) will enable us to solve many problems with great ease, as we will now show.

**282.** Consider Prob. I. Here $U = \int_{s_0}^{s_1} ds$, so that $v = 1$, $v_x = 0$, $v_y = 0$. Therefore equation (25) gives $\frac{1}{r} = 0$. Hence, $r$ being infinite, the solution must be a right line.

Turn next to Case 2, Prob. II. Here $U$ may be written $U = \int_{s_0}^{s_1} \frac{ds}{\sqrt{y}}$, so that $v = \frac{1}{\sqrt{y}}$, $v_x = 0$, $v_y = -\frac{1}{2y^{\frac{3}{2}}}$, and (25) gives $\frac{1}{r} = \frac{\cos B}{2y}$ and $r = 2y \sec B$. Let $n$ be the normal. Then $n = y \sec B$ and $r = 2n$, which is known to indicate that the required curve must be a cycloid.

In the last place, consider Prob. VII. Here we may write $U = \int_{s_0}^{s_1} y\,ds$, so that $v = y$, $v_x = 0$, $v_y = 1$, and (25) gives $\frac{1}{r} = -\frac{\cos B}{y}$ and $r = -y \sec B$. Hence, in this case, the radius of curvature must equal the normal estimated in an opposite direction, and this is known to indicate that the curve is a catenary, the directrix being the axis of $x$.

**283.** In all these problems we shall obtain the same equations at the limits for the determination of the arbitrary constants as we would if we had regarded $x$ as the independent variable. For suppose, first, the curve is to connect two fixed points. Then, as shown in Art. 279, the limiting terms

will take the form of (19), and $v$ and $l$ being always equal, they will entirely vanish, so that the constants must be determined by the circumstance that the curve is to pass through two fixed points, which are evidently the same conditions as we would have obtained had we assumed $x$ as the independent variable. If we next require that the curve shall always have its extremities upon two fixed curves whose equations are as in Art. 278, then we shall obtain equations (21). Now the first of these equations gives no direct condition regarding the limits, but, with the aid of the second, serves merely to determine $c$ in (16), $c$ being an additional constant necessarily introduced by the employment of the new quantity $l$. But dividing the second of these equations by $y_1'$, and multiplying by $(y_x)_1$, we find $(1 + f'y_x)_1 = 0$, and a similar equation for the lower limit. These equations show that the required curve must meet its limiting curves at right angles, which conditions are also the same as would have been obtained had we assumed $x$ as the independent variable.

## Problem XLVI.

**284.** *Let $v$ and $u$ be any functions of $x$ and $y$ only, with constants, and let it be required to maximize and minimize the expression*

$$U = \int_{x_0}^{x_1}\left(v\sqrt{1+y'^2} + u\right)dx = \int_{s_0}^{s_1}\left(v + ux'\right)ds = \int_{s_0}^{s_1} Vds. \quad (1)$$

Here, as before, because $s$ has been made the independent variable, $x$ and $y$, and consequently their variations, cannot be regarded as entirely independent. But equation (6), Art. 273, must always hold between $x$ and $y$; and as this gives an implicit relation between them, the variation of that equation must involve such a relation between their variations. Hence, multiplying the variation of (6), as before, by an unknown

quantity $l$, and transforming the variations by equations (8) and (10), we may, as before, write equation (11), Art. 274.

Now it will appear, by reasoning precisely like that employed in the last problem, that to vary $U$ in the most general manner, even when the required curve is to pass through two fixed points, we must add to the terms at the limits the terms $V_1 ds_1 - V_0 ds_0$. For it is evident that the reasoning there used would be equally applicable if, instead of supposing $v$ to be a function of $x$ and $y$ only, it had been any function of $x, y, x', x'', y', y''$, etc. Now varying (1), adding equation (11), Art. 274. and integrating by parts as usual, we shall obtain

$$\delta U = V_1 ds_1 - V_0 ds_0 + (u + lx')_1 \delta x_1$$

$$- (u + lx')_0 \delta x_0 + l_1 y_1' \delta y_1 - l_0 y_0' \delta y_0$$

$$+ \int_{s_0}^{s_1} \{[v_x + u_x x' - u' - (lx')'] \delta x + [v_y + u_y x' - (ly')'] \delta y\} ds$$

$$= L_1 - L_0 + \int_{s_0}^{s_1} \{M \delta x + N \delta y\} ds. \tag{3}$$

Here, as before, $L_1 - L_0$ and the integral must severally vanish whatever be the value of $l$.

If now, as before, we suppose $l$ to be such a quantity as will reduce $M$ or $N$ to zero throughout $U$, it will appear by the same reasoning as before that the other must vanish also. Making $M$ and $N$ zero in (3), we have

$$(lx' + u)' = v_x + u_x x', \qquad (ly')' = v_y + u_y x'. \tag{4}$$

Multiplying these equations respectively by $x'$ and $y'$, and adding, observing equation (6), Art. 273, we have

$$l' + u'x' = v_x x' + v_y y' + x'(u_x x' + u_y y') = v' + u'x'. \tag{5}$$

Hence, as before,

$$l' = v' \qquad \text{and} \qquad l = v + c. \tag{6}$$

**285.** Now in determining $c$ we must remember, as before, that if we can express $L_1 - L_0$ in terms of $ds_1$ and $ds_0$, we may, since these quantities are independent, equate their coefficients severally to zero; so that we need here consider but one limit. Let us first suppose that the curve is to pass through two fixed points. Then, taking the value of $L_1$ from (3), and substituting in it the values of $\delta x_1$ and $\delta y_1$ from equations (17), Art. 277, and remembering equation (6), Art. 273, we find $v_1 - l_1 = 0$, which shows, as before, that $v = l$ throughout $U$, $c$ in (6) being zero.

Next suppose the curve is to connect two fixed curves whose equations are as in Art. 278. Then in $L_1$ substitute the value of $\delta y_1$ found by transposing equation (18), Art. 278, and equate the coefficients of $ds_1$ and $\delta x_1$ severally to zero, because these quantities must be independent. Then, we shall have

$$u_1 + l_1 x_1' + l_1 y_1' f_1' = 0, \quad v_1 + u_1 x_1' + l_1 y_1' x_1' f_1' - l_1 y_1'^2 = 0. \quad (7)$$

Multiplying the first of these equations by $x_1'$ and subtracting the second from the product, we shall, by observing equation (6), Art. 273, have $l_1 - v_1 = 0$, so that here also $v = l$.

**286.** Putting $v$ for $l$, and differentiating the first term in each, equations (4) become

$$v_x - v'x' + u_x x' - u' = vx'', \qquad v_y - v'y' + u_y x' = vy''. \quad (8)$$

Now multiply the first term in each of these equations by $x'^2 + y'^2$, and put for $v'$ and $u'$ their values. Then factoring, we have

$$y'(v_x y' - v_y x' - u_y) = vx'', \qquad x'(v_y x' - v_x y' + u_y) = vy''. \quad (9)$$

Multiplying the first of these equations by $y'$, the second by $x'$, and subtracting the second from the first, we readily obtain, as before,

$$\frac{1}{r} = -\frac{1}{v}(v_x \cos A + v_y \cos B + u_y). \quad (10)$$

**287.** Let us next apply this formula to a few cases, beginning with Prob. XV. Here $U = \int_{s_0}^{s_1} (yx' + a)\,ds$, so that $v = a$, $v_x = 0$, $v_y = 0$, $u = y$, $u_y = 1$. Therefore equation (10) gives $\frac{1}{r} = -\frac{1}{a}$. Hence the curve must be a circle, since $r$ is a constant. The negative sign is in this case as it should be, because it has been shown that $a$ must be negative.

Turn next to Prob. XVI. Here $U = \int_{s_0}^{s_1} (y^2x' + ay)\,ds$; so that $v = ay$, $v_x = 0$, $v_y = a$, $u = y^2$, $u_y = 2y$; and equation (10) gives

$$\frac{1}{r} = -\frac{\cos B}{y} - \frac{2}{a}.$$

But $\frac{\cos B}{y} = \frac{1}{n}$, $n$ being the normal; and as we have already shown that $a$ must be negative, we may write $\frac{1}{r} + \frac{1}{n} = \frac{1}{A}$. We cannot in this case proceed to the solution obtained in Prob. XVI. without expressing the value of $r$ and integrating as in that problem, although it is evident enough that the sphere will satisfy the last equation.

We may remark, in passing, that Probs. XVII. and XVIII. are to be regarded as belonging to the preceding problem, because the factor of $ds$ is a function of $x$ and $y$ only, together with constants.

**288.** Here also the conditions for the determination of the two constants which will enter the complete integral of equation (10) will be always the same as though we had assumed $x$ as the independent variable. For if the curve must pass through two fixed points, we shall have for the upper limit

$$L_1 = (v_1 - l_1)ds_1 = (v_1 - v_1)ds_1.$$

That is, the limiting terms will vanish as they would by the other method. But suppose the curve is to connect two fixed curves. Then if $x$ were the independent variable, we would obtain for the upper limit

$$u_1(1+y'^2)_1^{\frac{1}{2}}+v_1(1+y_1'f_1')=0;$$

and multiplying by $x'$, remembering equation (6), Art. 273, we shall obtain the first of equations (7). Now the second of these equations gives no new condition, but merely enables us to determine the constant $c$ in (6). To ascertain these conditions, let $o$ be the angle between the required and the upper fixed curve at their intersection, $t$ the angle whose tangent is $f'$, and $a$ the angle whose cosine is $x'$. Then, multiplying the first of equations (7) by cos $t$, we have

$$u_1\cos t+v_1(\cos a\cos t+\sin a\sin t)=$$
$$u_1\cos t+v_1\cos o=0. \qquad (11)$$

### Problem XLVII.

**289.** *Let $r$ be the radius of curvature of a plane curve, and $V$ any function of $r$ and constants. Then it is required to determine the conditions which will maximize or minimize the expression*

$$U=\int_{s_0}^{s_1}Vds. \qquad (1)$$

Here

$$\delta U=V_1ds_1-V_0ds_0+\int_{s_0}^{s_1}V_r\delta r\,ds. \qquad (2)$$

Now the following equations are known to be true:

$$\left.\begin{array}{l}\dfrac{1}{r}=R=y'x''-x'y'', \qquad \dfrac{1}{r^2}=R^2=x''^2+y''^2, \\ x'^2+y'^2=1, \qquad x'\delta x'+y'\delta y'=0, \qquad x'x''+y'y''=0.\end{array}\right\} \qquad (3)$$

We must now obtain $\delta r$. We have

$$\delta(R^2)=2(x''\delta x''+y''\delta y'')=\frac{-2\delta r}{r^3}.$$

Whence $$\delta r = -r^3(x''\delta x'' + y''\delta y''). \tag{4}$$

Hence, proceeding as before, and putting $v$ for $V_r r^3$, we have

$$\delta U = V_1 ds_1 - V_0 ds_0$$
$$+\int_{s_0}^{s_1} \{-v(x''\delta x'' + y''\delta y'') + l(x'\delta x' + y'\delta y')\} ds = 0. \tag{5}$$

Whence, as usual, we obtain, after changing signs, the equations

$$(vx'')'' + (lx')' = 0, \qquad (vy'')'' + (ly')' = 0, \tag{6}$$

and

$$\left.\begin{aligned}(vx'')' + lx' &= a = vx''' + v'x'' + lx',\\ (vy'')' + ly' &= b = vy''' + v'y'' + ly'.\end{aligned}\right\} \tag{7}$$

Multiplying the first of these equations by $y'$, the second by $x'$, and subtracting the second from the first, we have

$$v(y'x''' - x'y''') + v'(y'x'' - x'y'') = ay' - bx' = vR' + Rv'. \tag{8}$$

Whence $$vR = V_r r^3 = ay - bx + c. \tag{9}$$

**290.** It will be seen that in this case $l$ has been eliminated, and we will now examine the method of determining the constants in (9). Consider the terms at the upper limit, arising from the usual transformation of (5). These are

$$V_1 ds_1 + \{(vx'')' + lx'\}_1 \delta x_1$$
$$+ \{(vy'')' + ly'\}_1 \delta y_1 - v_1(x''\delta x' + y''\delta y')_1 = 0. \tag{10}$$

Now it at once appears from (7) that the coefficients of $\delta x_1$ and $\delta y_1$ are respectively $a$ and $b$: and if for $\delta y'$ we put its value $\dfrac{-x'\delta x'}{y'}$ derived from the fourth of equations (3), the terms beyond $\delta y_1$ will become

$$-\left\{\frac{v(y'x'' - x'y'')}{y'}\right\}_1 \delta x_1' = -\left\{\frac{V_r r^2}{y'}\right\}_1 \delta x_1'. \tag{11}$$

Hence the terms at the upper limit become

$$V_1 ds_1 + a\delta x_1 + b\delta y_1 - \left\{ \frac{V_r r^2}{y'} \right\}_1 \delta x_1' = 0; \qquad (12)$$

and a similar equation will evidently hold at the lower limit.

Now the last term of the first member of (12) is evidently independent of the others, so that we must have $V_r r^2 = 0$ at both limits. Now suppose the line joining the extremities of the required curve be assumed as the axis of $x$. Then, because $y$ and $V_r r^2$ vanish at both limits, we have, from (9),

$$0 = -bx_0 + c \quad \text{and} \quad 0 = -bx_1 + c;$$

so that $b$ and $c$ must vanish, and then (9) becomes

$$V_r r^2 = ay. \qquad (13)$$

**291.** Suppose the curve is to pass through two fixed points. Then the terms at the upper limit become

$$V_1 ds_1 + a\delta x_1 = (v - ax')_1 ds_1 = 0,$$

the second member resulting from the elimination of $\delta x_1$ by means of equation (17), Art. 277; and a similar equation holds for the lower limit.

But suppose the curve is to connect two fixed curves whose equations are as heretofore. Then the terms at the upper limit are

$$V_1 ds_1 + a\delta x_1 + b\delta y_1 = 0. \qquad (14)$$

Eliminating $\delta y_1$ by means of equation (18), Art. 278, and then equating severally to zero the coefficients of $ds_1$ and $\delta x_1$, we shall obtain

$$V_1 + bf_1'x_1' - by_1' = 0, \qquad a + bf_1' = 0; \qquad (15)$$

and similar equations for the other limit. Now if the axis of $x$ join the points of intersection of the required curve and the

two fixed curves, $b$ will vanish, while $a$ cannot, as appears from equation (13); so that the second of equations (15) can only be satisfied by supposing $f_1'$ to be infinite.

Hence the tangents to the two fixed curves at their points of intersection with the required curve must be at right angles to the line joining those points.

**292.** As an example of the foregoing theory, consider Prob. III.

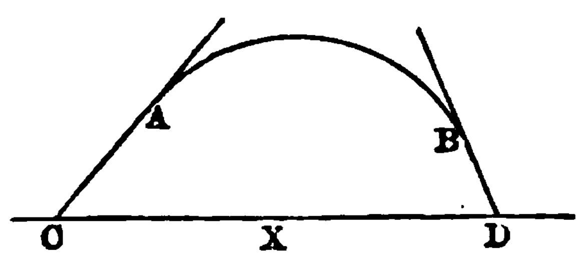
Here

$$U = \int_{x_0}^{x_1} \frac{(1 + y'^2)^2}{y''} dx = \int_{s_0}^{s_1} r ds;$$

so that $V = r$, $V_r = 1$, and equation (13) gives $r^2 = ay$. Now as the axis of $x$ in this case joins the two extremities of the required curve, it is readily seen that the cycloid having its cusps upon the axis of $x$ is a solution, because in such a cycloid $r = 2\sqrt{Dy}$, $D$ being the diameter of the generating circle.

**293.** Another interesting application is the following:

*An elastic spring AB is adjusted between two right lines so as to be tangent to both at its extremities A and B; it is required to determine the form which the spring must assume in order to be in equilibrium.*




According to the principle of Daniel Bernoulli, the curve $AB$ must be such as to minimize the expression $U = \int_{s_0}^{s_1} \frac{ds}{r^2}$.

Hence $V = \frac{1}{r^2}$, $V_r = \frac{-2}{r^3}$, and equation (9) becomes

$$\frac{-2}{r} = ay - bx + c. \qquad (16)$$

But since $AB$ is compelled to be tangent to the lines $AC$ and $BD$, its extreme tangents have a fixed inclination to the axis of $x$, and therefore $\delta x_1'$, $\delta y_1'$, $\delta x_0'$ and $\delta y_0'$ vanish, and we need not now have $V_r r^2 = 0$ at either limit. But equations (15) are universally true, and the second of these gives

$$a + bf' = 0 \qquad \text{and} \quad a + bF' = 0. \tag{17}$$

But since the lines $AC$ and $BD$ are not parallel, the constants $f'$ and $F'$, which are the tangents of the inclinations of these lines to the axis of $x$, are unequal; so that in this case we find that $a$ and $b$ must vanish. Then, by (16), we find that $r$ is a constant, so that $AB$ must be a circular arc if $r$ be finite.

But now the first of equations (15) would appear to give $V = \frac{1}{r^2} = 0$ for both limits; which evidently cannot be true. To obviate this difficulty we must suppose the spring to have a given length. Then $ds_1$ and $ds_0$ will vanish, and the first of equations (15) will not necessarily hold.

But under this supposition we should, according to Euler's method, have written $V = \frac{1}{r^2} + c'$, which would produce no change in any equation except the first of equations (15); and this, when $a$ and $b$ vanish, would give $\frac{1}{r^2} + c' = 0$ at either limit, which presents no difficulty.

*Third Method.*

**294.** We have already seen that when $x$ is the independent variable, we are, although the supposition is unnatural, permitted to vary $x$; and in like manner, when $s$ is the independent variable, we may ascribe variations to $s$ throughout the range of integration. Indeed, this is the method usually adopted; and as we are generally obliged to increase or decrease $s$ at its limits, the method does not seem altogether

unnatural. The following illustration may perhaps aid us in forming a better conception of the two methods.

**295.** Suppose we had a curve $AB$ connecting two fixed points or two fixed curves, and suppose the curve to be formed of non-elastic wire on which notches are placed at our pleasure, the wire extending somewhat beyond $A$ and $B$. Then when we vary the form of $AB$ in the most general manner consistent with variations, we shall, in general, find that we are unable to make the new curve connect the two points or curves without either adding or excluding certain wire adjacent to $A$ and $B$. Still the distance of any notch from some given notch—that is, $s$—undergoes no change, a positive or negative increment merely being added to the limits. This may illustrate what takes place in the first method.

Now suppose the original piece to be expanded by heat or contracted by cold until it is able to form the required arc of the derived curve. Then, although we increase or diminish the length of the arc $AB$, we do not add or exclude any wire. But now the distance of any notch from the given notch, or $s$, will have undergone an infinitesimal change; that is, will have become $s + \delta s$. But, to render the illustration complete, we must suppose the motion of any particular notch to be capable of taking either a positive or negative direction, or of becoming zero, or, in short, of following any law we please. In this case we would have an illustration of the method which we are now about to employ.

**296.** Let us now examine the mode of employing this method.

Assume the equation

$$U = \int_{s_0}^{s_1} V ds, \tag{1}$$

where $V$ is any function of $s$, $x$, $x'$, $x''$, .... $y$, $y'$, $y''$ .... Now when we vary $s$, $x$, $y$, etc., the reasoning in the begin-

ning of Art. 264 is rendered applicable to the present case by reading $s$ for $x$. Moreover, all the equations, including (6), will be true if for $x$ we substitute $s$ in the limits, the differentials and the variations. Beginning then with (6), we have

$$\delta U = \int_{s_0}^{s_1} \frac{V d\delta s}{ds}\, ds + \int_{s_0}^{s_1} \delta V ds. \tag{2}$$

But

$$\int_{s_0}^{s_1} \frac{V d\delta s}{ds}\, ds = V_1 \delta s_1 - V_0 \delta s_0 - \int_{s_0}^{s_1} V' \delta s\, ds, \tag{3}$$

where accents denote total differential coefficients, while literal suffixes will denote partial differential coefficients; so that

$$\begin{aligned} V' = {} & V_s + V_x x' + V_{x'} x'' + V_{x''} x''' + \text{etc.} \\ & + V_y y' + V_{y'} y'' + V_{y''} y''' + \text{etc.} \end{aligned} \tag{4}$$

Now, to the first order, we have

$$\begin{aligned} \delta V = {} & V_s \delta s + V_x \delta x + V_{x'} \delta x' + V_{x''} \delta x'' + \text{etc.} \\ & + V_y \delta y + V_{y'} \delta y' + V_{y''} \delta y'' + \text{etc.} \end{aligned} \tag{5}$$

Hence

$$\begin{aligned} \delta U = {} & V_1 \delta s_1 - V_0 \delta s_0 \\ & + \int_{s_0}^{s_1} \{ V_x \delta x + V_{x'} \delta x' + V_{x''} \delta x'' + \text{etc.} \\ & + V_y \delta y + V_{y'} \delta y' + V_{y''} \delta y'' + \text{etc.} \\ & - (V_x x' + V_{x'} x'' + V_{x''} x''' + \text{etc.} \\ & + V_y y' + V_{y'} y'' + V_{y''} y''' + \text{etc.}) \delta s \} ds. \end{aligned} \tag{6}$$

Now employing $\omega$ as before (Art. 265), let

$$\omega^x = \delta x - x' \delta s \qquad \text{and} \qquad \omega^y = \delta y - y' \delta s.$$

Then, by the same method as that employed in Art. 265, we obtain

$$\left.\begin{aligned}\delta x' &= (\omega^x)' + x''\delta s, \qquad & \delta x'' &= (\omega^x)'' + x'''\delta s, \qquad & &\text{etc.,}\\ \delta y' &= (\omega^y)' + y''\delta s, \qquad & \delta y'' &= (\omega^y)'' + y'''\delta s, \qquad & &\text{etc.}\end{aligned}\right\} \quad (7)$$

But these equations are of course, like those in Art. 265, true to the first order only. By the use of these equations, (6) becomes

$$\delta U = V_1\delta s_1 - V_0\delta s_0$$

$$+\int_{s_0}^{s_1}\{V_x\omega^x + V_{x'}(\omega^x)' + V_{x''}(\omega^x)'' + \text{etc.}$$

$$+ V_y\omega^y + V_{y'}(\omega^y)' + V_{y''}(\omega^y)'' + \text{etc.}\}ds. \quad (8)$$

Hence, by the usual transformation, and giving for brevity only the general form of the terms at the limits, we have

$$\delta U = V\delta s + (V_{x'} - V_{x''}' + \text{etc.})\omega^x + (V_{x''} - \text{etc.})(\omega^x)' + \text{etc.}$$

$$+ (V_{y'} - V_{y''}' + \text{etc.})\omega^y + (V_{y''} - \text{etc.})(\omega^y)' + \text{etc.}$$

$$+\int_{s_0}^{s_1}\{(V_x - V_{x'}' + V_{x''}'' - \text{etc.})\omega^x$$

$$+ (V_y - V_{y'}' + V_{y''}'' - \text{etc.})\omega^y\}ds. \quad (9)$$

**297.** But $\delta x$ and $\delta y$, and consequently $\omega^x$ and $\omega^y$, are not wholly independent, because, whether we vary $s$ or not, the equations

$$x'^2 + y'^2 = 1 \qquad \text{and} \qquad x'x'' + y'y'' = 0 \qquad (10)$$

must always hold throughout both the primitive and derived curve. If, therefore, we wish to maximize or minimize $U$, and for this purpose equate $\delta U$ to zero, we must, as before, in order

to obtain any available equations of condition, employ the method of Lagrange. Now from (10) we have

$$x'\delta x' + y'\delta y' = 0 = x'(\omega^x)' + x'x''\delta s + y'(\omega^y)' + y'y''\delta s$$
$$= x'(\omega^x)' + y'(\omega^y)' + (x'x'' + y'y'')\delta s = x'(\omega^x)' + y'(\omega^y)'. \quad (11)$$

Therefore, $l$ being an undetermined quantity, we may, as before, write

$$\int_{s_0}^{s_1} l\{x'(\omega^x)' + y'(\omega^y)'\}ds = 0.$$

Now transform this equation and add it to (9), and let $L$ denote the general form of the limiting terms $L_1 - L_0$, $M$ and $N$ being the respective coefficients of $\omega^x ds$ and $\omega^y ds$ under the integral sign. Then we shall have

$$L = V\delta s + (V_{x'} - V_{x''}' + \text{etc.} + lx')\,\omega^x + (V_{x''} - \text{etc.})\,(\omega^x)' + \text{etc.}$$
$$+ (V_{y'} - V_{y''}' + \text{etc.} + ly')\,\omega^y + (V_{y''} - \text{etc.})\,(\omega^y)' + \text{etc.}, \quad (12)$$

$$M = V_x - V_{x'}' + V_{x''}'' - \text{etc.} - (lx')', \quad (13)$$

$$N = V_y - V_{y'}' + V_{y''}'' - \text{etc.} - (ly')'. \quad (14)$$

Now it is evident that (13) and (14) are the same differential equations as we would have obtained had we followed the preceding method, and ascribed no variation to $s$, $l$ of course in each case being supposed to be so taken as to cause either $M$ or $N$ to vanish, so that the other will vanish also. Hence, since the general solution will have the same form as before, it will be necessary, in further comparing the two methods, to consider only the terms at the limits.

**298.** It may be observed, in the first place, that the general form of the limiting terms is the same by the two methods; $\delta s_1$, $\delta s_0$ and the $\omega$'s and their differential coefficients in the

second method replacing $ds_1$, $ds_0$ and the $\delta$'s in the first. It would appear, therefore, that we might safely assume that the same conditions at the limits could be ultimately obtained by the two methods. But as it has not been deemed necessary to consider the most general form of $V$ by the other method, it will, we presume, be sufficient to give $V$ the same degree of generality in this; that is, to show that in the three preceding problems the same equations at the limits are obtained by either method.

Suppose we make $V$ a function of $x$ and $y$ only; that is, apply this method to Prob. XLV. Then, by (12), we have, for the upper limit,

$$L_1 = V_1\delta s_1 + (lx'\omega^x)_1 + (ly'\omega^y)_1 = 0. \qquad (15)$$

Now suppose the curve is to pass through two fixed points. Then $\delta x_1$ and $\delta y_1$ vanish, because by this method $x_1$ and $y_1$ mean the co-ordinates of the actual extremities of the arc, although $\delta x_1$ need not vanish, as the arc may have undergone an alteration in length. Hence $(\omega^x)_1 = -x_1'\delta s_1$, $(\omega^y)_1 = -y_1'\delta s_1$, and (15) gives

$$L_1 = \{V - l(x'^2 + y'^2)\}_1 = 0; \qquad (16)$$

so that $V_1 = l_1$.

Next suppose the curve is to connect two fixed curves whose equations are as usual. In this case we shall have $\delta y_1 = f_1'\,\delta x_1$. Substituting this value in (15) and equating severally to zero the coefficients of $\delta s_1$ and $\delta x_1$, because these quantities are entirely independent, that of $\delta s_1$ will give the second and third members of (16), while that of $\delta x_1$ will give

$$(lx' + ly'f')_1 = 0.$$

This is the same as the second of equations (21), Art. 279, the interpretation of which is given in Art. 283.

**299.** Next consider Prob. XLVI. Here $V = v + ux'$, $v$ and $u$ being functions of $x$ and $y$ only, so that $V_{x'} = u$. Therefore (12) gives

$$(v + ux')_1\delta s_1 + (u + lx')_1(\omega^x)_1 + l_1 y_1'(\omega^y)_1 = 0. \qquad (17)$$

If now the curve is to pass through two fixed points, $\delta x_1$ and $\delta y_1$ will vanish, and putting for $\omega^x$ and $\omega^y$ their values, the coefficient of $\delta s_1$ will take the form of the second member of (16), which shows, as before, that $V = l$.

Next suppose the curve is to connect two fixed curves. Then we have $\delta y_1 = f_1'\delta x_1$. Now substitute in (17) the values of $\omega^x$ and $\omega^y$, and eliminate $\delta y_1$. Then, as $\delta s_1$ and $\delta x_1$ are independent, we must equate their coefficients severally to zero. That of $\delta s_1$ will, as before, assume the form given in (16), showing that $V = l$, while that of $\delta x_1$ will become

$$(u + lx' + ly'f')_1 = 0.$$

But this is the first of equations (7), Art. 285, which has been already considered in Art. 288.

**300.** In the last place, consider Prob. XLVII. Here we have

$$\delta U = \int_{s_0}^{s_1} \left\{ \frac{Vd\delta s}{ds} ds + \delta V ds \right\}$$

$$= V_1\delta s_1 - V_0\delta s_0 + \int_{s_0}^{s_1} (-V'\delta s + \delta V) ds. \qquad (1)$$

Now

$$V' = V_r r'. \qquad (2)$$

But employing the reasoning by which we obtained $\delta r$ in Art. 289, only putting for every $\delta$ an accent, we find

$$r' = -r^3(x''x''' + y''y''');\qquad (3)$$

and therefore, putting $v = V_r r^3$, we have

$$V' = -v(x''x''' + y''y'''). \qquad (4)$$

We also have

$$\delta V = V_r \delta r = -v(x''\delta x'' + y''\delta y''), \tag{5}$$

$\delta r$ having the same value as in equation (4), Art. 289, although $\delta x''$ and $\delta y''$ have not now the same values. Therefore, by substitution, (1) becomes

$$\delta U = V_1 \delta s_1 - V_0 \delta s_0$$

$$+ \int_{s_0}^{s_1} \{ -v(x''\delta x'' + y''\delta y'') + v(x''x''' + y''y''')\delta s \} ds$$

$$= V_1 \delta s_1 - V_0 \delta s_0 + \int_{s_0}^{s_1} \{ -v[(\omega^x)'' + (\omega^y)''] \} ds. \tag{6}$$

Hence, integrating and employing the method of Lagrange, we shall evidently obtain for the general solution the same differential equations as before (equations (6), Art. 289).

Now the terms at the upper limit will be

$$V_1 \delta s_1 + \{ (vx'')' + lx' \}_1 (\omega^x)_1 + \{ (vy'')' + ly' \}_1 (\omega^y)_1$$

$$- v_1 \{ x''(\omega^x)' + y''(\omega^y)' \}_1 = 0, \tag{7}$$

which is similar to equation (10), Art. 290.

But from equation (11), Art. 297, we see that we can eliminate $(\omega^y)'$ in the same manner as we did $\delta y'$ in Art. 290; and as equations (7), Art. 289, are obtained in the general solution, the terms at the limit will become

$$V_1 \delta s_1 + a(\omega^x)_1 + b(\omega^y)_1 - (V_r r^3)_1 (\omega^x)_1' = 0. \tag{8}$$

But since $(\omega^x)_1' = \delta x_1' - x_1''\delta s_1$, if $\delta x_1'$ be unrestricted, so will $(\omega^x)_1'$, and its coefficient must vanish; so that, as before, $b$ and $c$ become zero, and we have

$$V_1 \delta s_1 + a(\omega^x)_1 = 0. \tag{9}$$

Then if the extremities be fixed, $\delta x_1$ becomes zero, and we have, as before (Art. 291), $(V - ax')_1 = 0$. But if the extremities are to be upon two given curves, then the terms at the limits become

$$V_1 \delta s_1 + a(\omega^x)_1 + b(\omega^y)_1 = 0. \tag{10}$$

Now substitute in (10) the values of $(\omega^x)_1$ and $(\omega^y)_1$, and also for $\delta y_1$ the value $f_1' \delta x_1$. Then equating severally to zero the coefficients of $\delta s_1$ and $\delta x_1$, we shall have

$$V_1 - a x_1' - b y_1' = 0 \qquad \text{and} \quad a + b f_1' = 0. \tag{11}$$

Eliminating $a$ from the first of these equations by means of the second, it becomes

$$V_1 + b f_1' x_1' - b y_1' = 0.$$

But the last two equations are equations (15), Art. 291, and we have, therefore, the same conditions as formerly.

Thus we see that while the equations for the general solution given by the two methods are always necessarily the same, the limiting equations are also the same eventually, at least so far as we have carried our investigations.

## CHAPTER II.

### MAXIMA AND MINIMA OF SINGLE INTEGRALS INVOLVING TWO OR MORE DEPENDENT VARIABLES.

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### SECTION I.

*CASE IN WHICH THE VARIATIONS ARE UNCONNECTED BY ANY EQUATION.*

### Problem XLVIII.

**301.** *It is required to determine the curve of minimum length which can be drawn between two fixed points given at pleasure in space.*

Let $ds$ be an element of the required curve. Then since the curve is to be situated in space, and is no longer necessarily plane, we have

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}}\, dx = \sqrt{1 + y'^2 + z'^2}\, dx.$$

Therefore the expression to be minimized in this case is

$$U = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx = \int_{x_0}^{x_1} V dx. \qquad (1)$$

Now it is evident that here, as in Prob. I., we must compare the required curve with such as are drawn indefinitely near at every point; and it is also evident that by giving to $y'$ and

$z'$ indefinitely small variations, these variations being wholly unrestricted as to sign, we can make any infinitesimal change we please in the form of the primitive or required curve. Now if we change $y'$ into $y' + \delta y'$, and $z'$ into $z' + \delta z'$, while $x$ undergoes no change, the corresponding alteration in the length of the required curve will be $\delta U$; and the method of finding $\delta U$ in its untransformed state needs no explanation; that already given being perfectly general whatever be the quantities involved in $U$.

**302.** Therefore, to the second order, we have

$$\delta U = \int_{x_0}^{x_1} \left\{ \frac{y'}{\sqrt{1+y'^2+z'^2}} \delta y' + \frac{z'}{\sqrt{1+y'^2+z'^2}} \delta z' \right\} dx$$

$$+ \frac{1}{2} \int_{x_0}^{x_1} \left\{ \frac{1+z'^2}{\sqrt{(1+y'^2+z'^2)^3}} \delta y'^2 + \frac{1+y'^2}{\sqrt{(1+y'^2+z'^2)^3}} \delta z'^2 \right.$$

$$\left. - \frac{2y'z'}{\sqrt{(1+y'^2+z'^2)^3}} \delta y' \delta z' \right\} dx. \quad (2)$$

Now it needs no additional explanation to show that if $U$ is to become a minimum, the first integral in (2) must vanish, while the second must become invariably positive. Hence, to the first order, we have

$$\delta U = \int_{x_0}^{x_1} \left\{ \frac{y'}{\sqrt{1+y'^2+z'^2}} \delta y' + \frac{z'}{\sqrt{1+y'^2+z'^2}} \delta z' \right\} dx$$

$$= \int_{x_0}^{x_1} \frac{y'}{\sqrt{1+y'^2+z'^2}} \delta y' dx + \int_{x_0}^{x_1} \frac{z'}{\sqrt{1+y'^2+z'^2}} \delta z' dx = 0. \quad (3)$$

But since $z$ is also a function of $x$, we may put $z$, $z'$, $z''$, etc., for $y$, $y'$, $y''$, etc., in the reasoning of Art. 9. Then we shall find $\delta z^{(n)} = \dfrac{d^n \delta z}{dx^n}$. In like manner it is evident that when $x$

receives no variation, if we had any number of variables $y$, $z$, $u$, etc., all regarded as functions of $x$, the reasoning of Art. 9 would apply to each, and we would have

$$\delta y^{(n)} = \frac{d^n \delta y}{dx^n}, \qquad \delta z^{(n)} = \frac{d^n \delta z}{dx^n}, \qquad \delta u^{(n)} = \frac{d^n \delta u}{dx^n}, \quad \text{etc.}; \tag{4}$$

and these equations will hold whether $y$, $z$, $u$, etc., are independent, or are connected by some equation.

**303.** Therefore, transforming $\delta U$ in the usual manner, we have

$$\delta U = \left\{ \frac{y'}{\sqrt{1 + y'^2 + z'^2}} \right\}_1 \delta y_1 - \left\{ \frac{y'}{\sqrt{1 + y'^2 + z'^2}} \right\}_0 \delta y_0$$

$$+ \left\{ \frac{z'}{\sqrt{1 + y'^2 + z'^2}} \right\}_1 \delta z_1 - \left\{ \frac{z'}{\sqrt{1 + y'^2 + z'^2}} \right\}_0 \delta z_0$$

$$- \int_{x_0}^{x_1} \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2 + z'^2}} \delta y \, dx - \int_{x_0}^{x_1} \frac{d}{dx} \frac{z'}{\sqrt{1 + y'^2 + z'^2}} \delta z \, dx = 0$$

$$= h_1 \delta y_1 - h_0 \delta y_0 + H_1 \delta z_1 - H_0 \delta z_0 + \int_{x_0}^{x_1} \{ M \delta y + N \delta z \} dx, \tag{5}$$

where $M$ and $N$ are, as previously, total differential coefficients. But as the required curve is to pass through two fixed points, $\delta y$ and $\delta z$ must vanish at both limits, so that $\delta U$ will consist only of the terms under the integral sign in (5).

Now $\delta y$ and $\delta z$ are here entirely independent. For we may suppose the derived curve to be obtained by varying one of the quantities $y$ or $z$, the other undergoing no change whatever; or we may suppose it to be such as would require us to vary both. Hence, that $\delta U$ may vanish, the two integrals in (5) must severally vanish.

But both $\delta y$ and $\delta z$ are entirely in our power, and are each as unrestricted as is $\delta y$ in Prob. I. Therefore, to make both in-

tegrals necessarily vanish severally, we must have $M = 0$ and $N = 0$.

**304.** Equating $M$ and $N$ severally to zero, we shall obtain

$$\frac{y'}{\sqrt{1 + y'^2 + z'^2}} = c \qquad \text{and} \qquad \frac{z'}{\sqrt{1 + y'^2 + z'^2}} = c'. \qquad (6)$$

Now solving these equations by common algebraic methods for $y'$ and $z'$, we find both these quantities to be constants, say $a$ and $a'$ respectively. Whence, by a second integration, we find

$$y = ax + b \qquad \text{and} \quad z = a'x + b', \qquad (7)$$

the equations of the right line in space.

This is, of course, only a general trial solution, and to render it applicable in any particular case we must show, first, that real values can be obtained for the four arbitrary constants which it contains, and, second, that the terms of the second order in $\delta U$ become positive.

**305.** Let us first suppose that the line is to pass through two fixed points whose co-ordinates $x_1$, $y_1$, $z_1$ and $x_0$, $y_0$, $z_0$ are known. Then we have

$$y_1 = ax_1 + b, \quad y_0 = ax_0 + b, \quad z_1 = a'x_1 + b', \quad z_0 = a'x_0 + b', \quad (8)$$

and these equations are evidently sufficient for the determination of the constants $a$, $b$, $a'$ and $b'$; and we see that, because these constants have the meaning explained in works on analytical geometry, they will always have real values.

But suppose the limiting values of $x$ only to be fixed; that is, that the line is merely to have its extremities always situated in two fixed planes, each perpendicular to the axis of $x$, their equations being $x = x_0$ and $x = x_1$. Then it will appear, by the same reasoning as has been hitherto employed, that the portion of $\delta U$ remaining under the integral sign must be en-

tirely independent of that which is free from this sign. It must, moreover, be plain that the last statement would hold even should $V$ contain other dependent variables besides $z$, and will also hold whether these variables be functions of $x$ which are completely independent, or are in some manner connected.

**306.** Therefore, since $L_1 - L_0$ must always vanish, we must here have

$$h_1\delta y_1 - h_0\delta y_0 + H_1\delta z_1 - H_0\delta z_0 = 0. \tag{9}$$

Now in the present case it is evident that the quantities $\delta y_1$, $\delta y_0$, $\delta z_1$, $\delta z_0$ are entirely independent, and hence the coefficients of these quantities must severally vanish, and we have

$$h_1 = 0, \qquad h_0 = 0, \qquad H_1 = 0, \qquad H_0 = 0. \tag{10}$$

But we see from (6) that $h = c$ and $H = c'$, so that (7) gives $y = b$ and $z = b'$, $a$ and $a'$ becoming zero. As the four conditions given by (10) are here equivalent to but two, the constants $b$ and $b'$ are undetermined. This case is similar to that in Art. 43, the line being here also parallel to $x$. If we fix the values of $y$ and $z$ at either limit, $b$ and $b'$ are determined, becoming those values respectively; and if we give one limiting value only, the constant which equals that value is determined, while the other remains undetermined.

**307.** It being possible to cause the terms of the first order in $\delta U$ to vanish, let us next consider whether those of the second order will become positive. Now it appears from (5) that these terms may be written

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1} \frac{\delta y'^2 + \delta z'^2 + z'^2\delta y'^2 - 2y'z'\delta y'\delta z' + y'^2\delta z'^2}{(1 + y'^2 + z'^2)^{\frac{3}{2}}}\,dx$$

$$= \frac{1}{2}\int_{x_0}^{x_1} \frac{\delta y'^2 + \delta z'^2 + (z'\delta y' - y'\delta z')^2}{(1 + y'^2 + z'^2)^{\frac{3}{2}}}\,dx; \tag{11}$$

and as we may regard $(1+y'^2+z'^2)^{\frac{3}{2}}$ as positive, $\delta U$ is positive, and the solution renders $U$ a minimum.

**308.** Let us now consider the case in which the limiting values of $x$ also are to undergo variation. Here no new principle is involved. For, by the same reasoning as before, it must be evident that if $V$ be any function of $x, y, z, u$, etc., and the differential coefficients of $y, z, u$, etc., with respect to $x$, all being regarded as functions of $x$, and we change the limits into $x_1 + dx_1$, and $x_0 + dx_0$, and also vary all the quantities except $x$, and then approximate as before to the second order, we shall merely be obliged to add to the value of $\delta U$ obtained by supposing the limiting values of $x$ only to be fixed, the terms

$$V_1 dx_1 - V_0 dx_0 + \frac{V_1'}{2} dx_1^2 + \delta V_1 dx_1, \tag{12}$$

where accents denote total differentials, so that

$$\left.\begin{aligned} V' &= V_x + V_y y' + \text{etc.} + V_z z' + \text{etc.}, \\ \text{and} \quad \delta V &= V_y \delta y + V_{y'} \delta y' + \text{etc.} + V_z \delta z + V_{z'} \delta z' + \text{etc.} \end{aligned}\right\} \tag{13}$$

Therefore, if in the present case we regard the terms of the first order only, we must merely add to the limiting terms already obtained, the terms

$$V_1 dx - V_0 dx_0 \text{ or } \sqrt{(1+y'^2+z'^2)}_1 dx_1 - \sqrt{(1+y'^2+z'^2)}_0 dx_0.$$

But it is plain that if $dx_1$ and $dx_0$ be entirely unrestricted, we must have $\sqrt{1+y'^2+z'^2} = 0$ at both limits, which is clearly impossible without rendering $y'$ or $z'$ imaginary.

**309.** But suppose the required line is always to have its extremities upon two surfaces whose equations are known. Then it is plain that the quantities $\delta y_1$, $\delta z_1$ and $dx_1$ will not be entirely independent, although any two of them will be in-

dependent, so that if we can eliminate any one of the three, we shall have the same number of limiting equations as when $x_1$ and $x_0$ are fixed. We must, however, in this case adopt a method somewhat different from that by which we obtained equations (10), Art. 69.

As the most convenient form, let the equations of the surfaces at the upper and lower limits be respectively

$$f(x, y, z) = 0 = f \qquad \text{and} \qquad F(x, y, z) = 0 = F. \tag{14}$$

Considering the upper limit, suppose the required line when a minimum to meet the surface at a point $D$ before, and at a point $E$ after, having been varied. Also let the co-ordinates of $E$—which is, of course, indefinitely near $D$—be $x_1 + dx_1$, $Y_1$, and $Z_1$. Then, when in $f_1$ we substitute for the co-ordinates of $D$ those of $E$, we cause $f_1$ to undergo no change, as it will remain zero. But we can evidently pass from $D$ to $E$ by first passing to the derived curve without changing the value of the abscissa $x_1$, and then tracing along this curve until we reach a point whose abscissa is $x_1 + dx_1$, which must, by the conditions of the question, be the point $E$. Now by the first movement we, in $f_1$, change $y$ into $y + \delta y$, and $z$ into $z + \delta z$, thereby probably increasing or diminishing $f_1$, while by the second we, in the new value of $f_1$, change $x_1$ into $x_1 + dx_1$, which reduces $f_1$ again to zero.

We see, then, that if we change $y_1$ into $y_1 + \delta y_1$, $z_1$ into $z_1 + \delta z_1$, and $x_1$ into $x_1 + dx_1$, the increment which will result to $f_1$ will be zero. We have then, to the first order,

$$(f_y \delta y)_1 + (f_z \delta z)_1 + (f_x + f_y y' + f_z z')_1 dx_1 = 0. \tag{15}$$

This equation is true to the first order only, since the complete increment which $f_1$ would receive is absolutely zero, while we have merely obtained that increment to the first order. But we can obtain an equation true to the second order by merely developing (15) to the second order, and equating this development to zero.

**310.** To employ (15) in the present case write

$$f_{\prime} = \frac{f_y}{f_x} \qquad \text{and} \qquad f_{\prime\prime} = \frac{f_z}{f_x}. \tag{16}$$

Then, observing that here $y' = a$ and $z' = a'$, we have, from (15),

$$(1 + af_{\prime 1} + a'f_{\prime\prime 1})dx_1 + f_{\prime 1}\delta y_1 + f_{\prime\prime 1}\delta z_1 = 0. \tag{17}$$

We also have

$$L_1 = V_1\,dx_1 + h_1\delta y_1 + H_1\delta z_1 = 0.$$

Substituting the values of $V_1$, $h_1$, $H_1$, $y'$ and $z'$, and clearing fractions, we have

$$(1 + a^2 + a'^2)dx_1 + a\delta y_1 + a'\delta z_1 = 0.$$

Then substituting in the last equation the value of $dx_1$ derived from (17), clearing fractions, and equating severally to zero the coefficients of $\delta y_1$ and $\delta z_1$, we have, after changing sign,

$$\left.\begin{aligned} f_{\prime 1}(1 + a^2 + a'^2) - a(1 + af_{\prime 1} + a'f_{\prime\prime 1}) = 0, \\ f_{\prime\prime 1}(1 + a^2 + a'^2) - a'(1 + af_{\prime 1} + a'f_{\prime\prime 1}) = 0. \end{aligned}\right\} \tag{18}$$

Multiplying the first equation by $a'$, the second by $a$, and subtracting, we obtain

$$a'f_{\prime 1} - af_{\prime\prime 1} = 0. \tag{19}$$

But, by reduction, equations (18) become

$$f_{\prime 1} - a + f_{\prime 1}a'^2 - a'af_{\prime\prime 1} = f_{\prime 1} - a + a'(f_{\prime 1}a' - af_{\prime\prime 1}) = 0,$$

$$f_{\prime\prime 1} - a' + f_{\prime\prime 1}a^2 - aa'f_{\prime 1} = f_{\prime\prime 1} - a' - a(f_{\prime 1}a' - af_{\prime\prime 1}) = 0.$$

Hence, from (19), we see that $f_{\prime 1} = a$ and $f_{\prime\prime 1} = a'$; and it is clear that we can discuss the lower limit in a similar manner, so that

$$f_{\prime 1} = a, \qquad f_{\prime\prime 1} = a', \qquad F_{\prime 0} = a, \qquad F_{\prime\prime 0} = a'. \tag{20}$$

Now in determining the four constants $a$, $a'$, $b$ and $b'$, we shall be concerned with ten unknown quantities, $x_1$, $y_1$, $z_1$, $x_0$, $y_0$, $z_0$, $a$, $a'$, $b$ and $b'$. But we have, in addition to the four equations (20), the following six equations:

$$y_1 = ax_1 + b, \qquad z_1 = a'x_1 + b', \qquad f_1 = 0,$$

$$y_0 = ax_0 + b, \qquad z_0 = a'x_0 + b', \qquad F_0 = 0;$$

and it is evident, therefore, without going into the discussion of any particular case, that these ten equations are sufficient for the determination of all the quantities involved.

Now from (14) we have, for the upper limit,

$$f_x dx + f_y dY + f_z dZ = 0,$$

or

$$1 + f_{,} Y' + f_{,,} Z' = 0 = 1 + aY' + a'Z'; \qquad (21)$$

and since we may regard $Y'$ and $Z'$ as belonging to any right line drawn through $D$, and also lying in the tangent plane to the upper limiting surface at $D$, the required curve must be normal to any such line, and consequently to the tangent plane. Therefore, since similar equations would hold for the lower limit, we conclude that the required straight line must be normal to the given surfaces.

If, instead of surfaces, the straight line is to have its extremities upon two curves, let the equations of the upper curve be $dy = f'dx$ and $dz = F'dx$. Then, by reasoning like that in Art. 69, we shall find

$$\delta y_1 = (f' - y')_1 dx_1 \qquad \text{and} \qquad \delta z_1 = (F' - z')_1 dx_1;$$

and recollecting that $y' = a$, $z' = a'$, we shall obtain, by substituting these values in the most general form of $L_1 - L_0$, the equation $1 + af' + a'F' = 0$, together with a similar equation for the lower limit, so that the line must be normal to the two limiting curves.

It is evident, however, that in these latter cases, in which the limiting values of $x$ are not fixed, the results would be sometimes maxima and sometimes minima; and we must therefore repeat the caution frequently given heretofore—not to receive as final any results obtained by an examination of the terms of the first order alone.

## Problem XLIX.

**311.** *It is required to determine the curve in free space down which a particle, influenced by gravity alone, would descend from one fixed point, curve, or surface to another fixed point, curve, or surface in a minimum time.*

Assume the axis of $x$ vertically downward; and if the particle be supposed to have an initial velocity at the upper point, which is the lower limit of integration, let $h'$ be the height due to that velocity. Then the velocity at any point will be $\sqrt{2g(x+h')}$. Hence, in this case, we must minimize the expression

$$U=\int_{x_0}^{x_1}\frac{\sqrt{1+y'^2+z'^2}\,dx}{\sqrt{x+h'}}. \tag{1}$$

Now varying $y$ and $z$ as before, transforming the terms of the first order until they assume the form

$$\delta U=L_1-L_0+\int_{x_0}^{x_1}M\,\delta y\,dx+\int_{x_0}^{x_1}N\,\delta z\,dx, \tag{2}$$

and then equating $M$ and $N$ severally to zero, we have

$$\left.\begin{aligned} M&=-\frac{d}{dx}\frac{y'}{\sqrt{(x+h')(1+y'^2+z'^2)}}=0,\\ N&=-\frac{d}{dx}\frac{z'}{\sqrt{(x+h')(1+y'^2+z'^2)}}=0. \end{aligned}\right\} \tag{3}$$

$$\frac{y'}{\sqrt{(x+h')(1+y'^2+z'^2)}}=c, \quad \frac{z'}{\sqrt{(x+h')(1+y'^2+z'^2)}}=c'. \quad (4)$$

Now dividing the first of equations (4) by the second, we find that $\frac{y'}{z'}$ must be constant. This is sufficient to show that the curve required must be a plane curve, and hence we know that the solution must be a cycloid.

Thus we see that we can sometimes avoid the necessity of integrating completely the equations $M=0$ and $N=0$, by showing that the problem can be reduced to one of two co-ordinates; and indeed we could evidently have done the same thing in the preceding problem. But when we come to consider the terms of the second order we must evidently resume three co-ordinates, because we now require that the primitive curve shall be compared with all curves which can be derived from it by infinitesimal changes in $y$, $z$, $y'$, $z'$, etc., some of which may not be plane curves, and would be, therefore, excluded by the employment of two co-ordinates only. But if we compare the form of $V$ in this and the preceding problem, observing that $x+h'$ has no variation, and denote by $S$ the coefficient of $dx$ under the integral sign in equation (11), Art. 307, it will at once appear that these terms must become

$$\delta U=\frac{1}{2}\int_{x_0}^{x_1}\frac{S\,dx}{\sqrt{x+h'}},$$

which must be also essentially positive, since $\sqrt{x+h'}$ is positive throughout $U$.

**312.** Now since we know that the required cycloid must have the line joining its cusps parallel to the horizontal plane $xy$, and itself be in a vertical plane, its general equation need involve but five constants. For we have to consider the three co-ordinates of one of the cusps, the angle which

the line joining these cusps makes with the plane of $xy$, and, lastly, the radius of the generating circle. If we suppose $h$ zero, and the cusps to lie in the plane of $yz$, these constants will be reduced to four. But we have, as the most general form of the terms at the limits,

$$L_1 - L_0 = V_1 dx_1 - V_0 dx_0 + h_1 \delta y_1 - h_0 \delta y_0 + H_1 \delta z_1 - H_0 \delta z_0 = 0;$$

so that it appears, as before, that if the limiting values of $x$ be fixed, we shall have just the requisite number of conditions for the determination of the constants; and that if these limiting values be not fixed, we must restrict $dx_1$ and $dx_0$. If $h'$ be not zero, it is at once determined by the initial velocity; but we have only shown that the cycloid gives a minimum when the limiting values of $x$ are fixed.

**313.** Let us now consider briefly how many constants will occur in the general solution of problems of this class, and what are our means for determining them, as these are the only points which need any additional general explanation.

Assume the equation $U = \int_{x_0}^{x_1} V dx$, where $V$ is any function of $x, y, z, y', \ldots. y^{(n)}, z', \ldots. z^{(m)}$. Then, proceeding in the usual way, we obtain for the general solution the two differential equations $M = 0$ and $N = 0$. Now $M$ is of the order $2n$ in $y$, and $n + m$ in $z$, and $N$ is of the order $2m$ in $z$, and $m + n$ in $y$. Differentiating $N$ $2m$ times, and $M$ $n + m$ times, we shall have, together with $M$ and $N$, $3m + n + 2$ differential equations; the highest differentials involved in any of these equations being $2(m + n)$ in $y$, and $3m + n$ in $z$. Now eliminating $z$ and its $3m + n$ differential coefficients, we shall obtain an equation in $x$ and $y$ only, and the differential coefficients of $y$ with respect to $x$. The order of this equation must be $2(m + n)$, and its complete integral must therefore involve $2(m + n)$ arbitrary constants, which is the number which must be contained by the general solution.

Now if we examine the most general form of the limiting terms $L_1 - L_0$, it will at once appear that, unless some restriction be imposed, there must be as many independent terms as there are quantities $dx_1$, $dx_0$, $\delta y_1$, $\delta y_0$, $\delta y_1'$, $\delta y_0'$, .... $\delta y_1^{(n-1)}$, $\delta y_0^{(n-1)}$, $\delta z_1$, $\delta z_0$, .... $\delta z_1^{(m-1)}$, $\delta z_0^{(m-1)}$, the number of which will be $2(m+n)+2$, or merely $2(m+n)$, if the limiting values of $x$ be fixed, or if $dx_1$ and $dx_0$ be restricted as formerly. Moreover, it will appear, as before, that any condition which causes one of these equations to disappear will itself furnish a new equation of condition, so that the number of limiting equations will still remain equal to that of the arbitrary constants.

Nevertheless it is easy to see that the reasoning here employed may be subject to exceptions similar to those which have been explained in the case of two co-ordinates; but these will give the reader no serious difficulty.

**314.** We may now consider, as being somewhat connected with our subject, the principle of least, or more properly minimum action, particular cases of which have been already discussed.

### Problem L.

*A particle is to move in space from one fixed point to another, its motion being controlled solely by a system of incessant forces. Then $x$, $y$, and $z$ being the co-ordinates of any point of its path, $ds$ an element of this path, and $v$ the velocity of the particle at the end of any time $t$, it is required to show that the nature of this path must be such as to render $\delta U$ to the first order zero, where*

$$U = \int_{s_0}^{s_1} v\,ds = \int_{x_0}^{x_1} v\sqrt{1+y'^2+z'^2}\,dx.$$

Denoting by $X$, $Y$ and $Z$, as usual, the aggregated components of all the forces in the direction of the axes of $x$, $y$,

and $z$ respectively, we shall assume the well-known equation in mechanics,

$$v\,dv = \frac{1}{2}\,d(v^2) = X\,dx + Y\,dy + Z\,dz. \qquad (1)$$

Now if we suppose the particle to be moving along the required path, the symbol $d$, as applied to any quantity, denotes the change which that quantity undergoes when the independent variable, which we may here assume to be $x$, receives an infinitesimally small increment, the curve remaining unchanged. But if we draw any derived curve, and suppose the particle could pass from any point $p$ on the primitive to some point $P$ indefinitely near $p$, but on the derived curve, then if we give to the symbol $d$ the meaning already explained, we may denote by $\delta$ the corresponding change which the various quantities would undergo if the particle could pass from $p$ to $P$.

Now in passing from $p$ to $P$, just as in passing along any element of the primitive curve, we may assume that $X$, $Y$, and $Z$ remain constant; and hence, denoting by $v^x$, $v^y$ and $v^z$ the components of $v$ in the direction of $x$, $y$ and $z$ respectively, if we add to $x$, $y$, or $z$ any infinitesimal increment, the corresponding change in $\frac{(v^x)^2}{2}$, etc., would be $X$, $Y$, or $Z$ multiplied by those increments respectively, whether those increments were added as differentials for the purpose of enabling us to pass from one point to another on the primitive curve, or as variations for the purpose of enabling us to pass from any point on the primitive curve to an adjacent point lying on some derived curve.

But we have seen that any derived curve can be obtained without varying $x$, and we shall therefore consider $p$ and $P$ as having the same abscissa $x$. Hence, accents below denoting differentiation with respect to $t$, and those above with respect to $x$, and remembering that $Y = y_{,,}$ and $Z = z_{,,}$, (1) may be written

$$\frac{1}{2}\,\delta\,(v^2) = v\,\delta v = Y\delta y + Z\delta z = y_{,,}\delta y + z_{,,}\delta z. \qquad (2)$$

But the last member of (2) equals

$$(y_{,}\delta y + z_{,}\delta z)_{,} - (y_{,}\delta y_{,} + z_{,}\delta z_{,}). \qquad (3)$$

Now we have

$$v^2 = \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt} + \frac{dz^2}{dt^2} = x_{,}^2 + y_{,}^2 + z_{,}^2. \qquad (4)$$

Then varying $v^2$ under the supposition that neither $dt$ nor $dx$ undergoes any change from variations, we shall obtain

$$v\,\delta v = y_{,}\delta y_{,} + z_{,}\delta z_{,}. \qquad (5)$$

That this supposition may be made will appear if we remember that, in passing from $p$ to $P$, $v$ undergoes no change, so that $dx$ and $dt$ for that element of the curve maintain to each other whatever ratio they had before the curve was varied. Of course if we divide the whole time $t$ into equal parts $dt$, the corresponding differentials of $x$ cannot be supposed to be equal among themselves; but this inequality can in no way affect our problem.

Hence, admitting the validity of (5), equation (3) becomes

$$(y_{,}\delta y + z_{,}\delta z)_{,} - v\,\delta v,$$

and (2) may therefore be written

$$2v\delta v = \delta(v^2) = (y_{,}\delta y + z_{,}\delta z)_{,}. \qquad (6)$$

But since $v = \frac{ds}{dt}$, we have $v^2 = \frac{vds}{dt}$. Hence (6) gives, after clearing fractions,

$$\delta(vds) = d(y_{,}\delta y + z_{,}\delta z). \qquad (7)$$

But since the particle is to pass from one fixed point to another, the derived curve must also pass through these

points, and we are not to suppose the particle capable of any displacement at either point, so that the variations of $y$ and $z$ vanish at both these points. Moreover, although we have really regarded $t$ as the independent variable, we may integrate (7) as though that variable were $x$. For $d$ in (7) denotes the change which $y_{\prime}\delta y + z_{\prime}\delta z$ undergoes in the time $dt$, or while the particle passes from a point whose abscissa is $x$ to one whose abscissa is $x + dx$; so that it is the same thing whether we suppose these changes to be summed up for the time $t_1 - t_0$ or through the distance $x_1 - x_0$. Therefore, by integration, (7) gives

$$\int_{s_0}^{s_1} \delta(vds) = \int_{x_0}^{x_1} \delta\left(v\sqrt{1 + y'^2 + z'^2}\right) dx$$

$$= (y_{\prime}\delta y + z_{\prime}\delta z)_1 - (y_{\prime}\delta y + z_{\prime}\delta z)_0 = 0. \qquad (8)$$

**315.** To guard against certain misconceptions, we observe, first, that the reasoning here employed would not be applicable if the particle were compelled by a system of forces to move along a fixed material curve. For then, although equation (1) would hold, equation (2) would not, because that portion of $X$, $Y$ and $Z$ which arises from the normal pressure of the curve upon the particle would vanish for any point $P$ without the curve; so that we could not say, as formerly, that, in passing from $p$ to $P$, $X$, $Y$, and $Z$ would remain constant.

We observe, secondly, that although the principle just established is commonly called that of least or minimum action, the name is not warranted, at least by the preceding demonstration. For our approximations were carried to the first order only; so that we are merely able to say that the required curve must be such as to render $\delta U$ to the first order zero. But we have already seen that the terms of the second order in $\delta U$ do not always become positive, but sometimes vanish also, in which cases we inferred, although we did not investigate the matter, that those of the third order would not likewise vanish, and that therefore $\delta U$ might have either sign

at our pleasure, thus showing that $U$ could be neither a maximum nor a minimum. It will be found, however, that the terms of the second order in $\delta U$ never become negative, and indeed it is generally conceded that the action can never, as Lagrange erroneously supposed, become a maximum.

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## SECTION II.

### CASE IN WHICH THE VARIATIONS ARE CONNECTED BY EQUATIONS, DIFFERENTIAL OR OTHER.

### Problem LI.

**316.** *It is required to determine the nature of the line of minimum length which can be drawn between two fixed points or curves on the surface of a sphere.*

Here

$$U = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx = \int_{x_0}^{x_1} V dx;$$

and taking the variation of $U$, and integrating in the usual manner, we have

$$\delta U = \left\{ \frac{y'}{\sqrt{1 + y'^2 + z'^2}} \right\}_1 \delta y_1 - \left\{ \frac{y'}{\sqrt{1 + y'^2 + z'^2}} \right\}_0 \delta y_0$$

$$+ \left\{ \frac{z'}{\sqrt{1 + y'^2 + z'^2}} \right\}_1 \delta z_1 - \left\{ \frac{z'}{\sqrt{1 + y'^2 + z'^2}} \right\}_0 \delta z_0$$

$$+ \int_{x_0}^{x_1} - \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2 + z'^2}} . \delta y\, dx + \int_{x_0}^{x_1} - \frac{d}{dx} \frac{z'}{\sqrt{1 + y'^2 + z'^2}} . \delta z\, dx$$

$$= h_1 \delta y_1 - h_0 \delta y_0 + H_1 \delta z_1 - H_0 \delta z_0$$

$$+ \int_{x_0}^{x_1} M \delta y\, dx + \int_{x_0}^{x_1} N \delta z\, dx = 0. \qquad (1)$$

Now in this case the variations of $y$ and $z$ are not independent, all derived curves which cannot be drawn upon the surface of the given sphere being excluded from comparison with the primitive. Nevertheless it is evident that the integrated and the unintegrated parts of $\delta U$ must severally vanish. For we may suppose each part to be expressed in terms containing one variation only, the other having been eliminated. Hence we have

$$L_1 - L_0 = 0, \qquad \int_{x_0}^{x_1}(M\delta y + N\delta z)\,dx = 0. \tag{2}$$

But we have from the sphere

$$x^2 + y^2 + z^2 = r^2, \qquad y\delta y + z\delta z = 0, \qquad \delta z = \frac{-y\delta y}{z}. \tag{3}$$

Hence (2) may now be written

$$\int_{x_0}^{x_1}\left\{M - \frac{Ny}{z}\right\}\delta y\,dx = \int_{x_0}^{x_1} M'\delta y\,dx = 0. \tag{4}$$

Whence it will at once appear that to maximize or minimize $U$, $M'$ must vanish. Equating $-M'$ to zero, we have

$$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2+z'^2}} - \frac{d}{dx}\frac{z'}{\sqrt{1+y'^2+z'^2}}\cdot\frac{y}{z} = 0,$$

so that we obtain

$$\frac{1}{y}\,d\frac{y'}{\sqrt{1+y'^2+z'^2}} = \frac{1}{z}\,d\frac{z'}{\sqrt{1+y'^2+z'^2}}. \tag{5}$$

**317.** Before proceeding, we shall find it necessary to change the independent variable to $s$. It is evident that (5) may be written

$$\frac{1}{y}\,d\frac{dy}{ds} = \frac{1}{z}\,d\frac{dz}{ds}. \tag{6}$$

Although the symbol $d$ in (6) denotes change incident upon changes in $x$, yet we were not originally bound to consider two consecutive values of $dx$ as absolutely equal, and we may therefore suppose that these differentials were so taken as to make those of $s$ always equal. Hence, regarding $ds$ as always constant, multiplying by $\frac{1}{ds}$, and denoting by accents differentiation with respect to $s$, (6) becomes

$$\frac{y''}{y} = \frac{z''}{z}. \tag{7}$$

Now multiply both the numerator and denominator of the first fraction by $y'$, and of the second by $z'$, and denote the resulting fractions by $\frac{A}{B}$ and $\frac{C}{D}$, which are, of course, equal to each other and to the members of (7). Hence the quantities $A$, $B$, $C$ and $D$ are in proportion, and therefore

$$A + C : B + D :: A : B :: C : D :: y'' : y :: z'' : z.$$

Hence either member of (7) equals

$$\frac{A + C}{B + D} = \frac{y'y'' + z'z''}{yy' + zz'}. \tag{8}$$

But from the equation of the sphere, and also the equation $x'^2 + y'^2 + z'^2 = 1$, we have

$$xx' + yy' + zz' = 0, \qquad x'x'' + y'y'' + z'z'' = 0:$$

and therefore (8) becomes $\frac{x''}{x}$, so that we have

$$\frac{x''}{x} = \frac{y''}{y} = \frac{z''}{z}. \tag{9}$$

Now because (9) is true we may evidently write the following three equations:

$$xd^2y - yd^2x = 0, \qquad xd^2z - zd^2x = 0, \qquad yd^2z - zd^2y = 0.$$

Integrating these equations by parts, we obtain

$$xdy - ydx = a', \qquad xdz - zdx = b', \qquad ydz - zdy = c'. \quad (10)$$

Multiplying the first of these equations by $z$, the second by $-y$, the third by $x$, and adding the products, we shall obtain

$$a'z - b'y + c'x = 0. \quad (11)$$

In this equation the constants are infinitesimal, but dividing by one of them, as $c'$, the two resulting constants may have any value we please, infinitesimal, finite, or infinite; and we may write $x + ay + bz = 0$, the equation of a plane passing through the centre of the sphere. The required curve must, therefore, be a great circle.

**318.** To determine the constants $a$ and $b$, we have, if we suppose the curve to connect two fixed points, the equations

$$x_1 + ay_1 + bz_1 = 0, \qquad x_0 + ay_0 + bz_0 = 0.$$

But suppose the curve is to connect two given curves, and let the equations of the curve for the upper limit be

$$y = f(x) = f, \quad z = F(x) = F, \quad \text{or} \quad dy = f'dx, \quad dz = F'dx. \quad (12)$$

Then we shall have, as in Art. 69,

$$\delta y_1 = (f' - y')_1 dx_1 \qquad \text{and} \qquad \delta z_1 = (F' - z')_1 dx_1. \quad (13)$$

Also it is evident, as before, that $L_1$ and $L_0$ must severally vanish, and

$$L_1 = \sqrt{(1 + y'^2 + z'^2)}_1\, dx_1$$

$$+ \left\{ \frac{y'}{\sqrt{1 + y'^2 + z'^2}} \right\}_1 \delta y_1 + \left\{ \frac{z'}{\sqrt{1 + y'^2 + z'^2}} \right\}_1 \delta z_1 = 0. \quad (14)$$

Eliminating $\delta y_1$ and $\delta z_1$ by means of (13), and reducing, we obtain $1 + y_1' f_1' + z_1' F_1' = 0$, which shows that the great circle must cut the limiting curve at right angles; and a similar result can evidently be obtained for any curve at the lower limit.

If we suppose the limiting values of $x$ only to be fixed, we shall obtain for either limit, after having eliminated $\delta z$ by (3), $zdy - ydz = 0 = -c'$. Hence (11) becomes

$$a'z - b'y = 0 = z - a''y, \qquad \text{or} \quad z = a''y,$$

where $a''$ remains undetermined, as it should, it being the tangent of the inclination of the great circle to the plane of $xy$. We conclude, therefore, that the great circle must be so drawn that its intersection with the plane of $xy$ shall always coincide with the axis of $x$.

**319.** It will appear by reference that $U$ has here the same general form as in the first problem of this chapter; and hence if the limiting values of $x$ be fixed, the terms of the second order, as they at first arise, will be the same as in equation (11), Art. 307, which may be written $\delta U = \frac{1}{2}\int_{x_0}^{x_1} S dx$. But now the mode of eliminating $\delta z$ must be rendered exact to the second order, and for this purpose we have

$$\delta z = \frac{dz}{dy}\delta y + \frac{1}{2}\frac{d^2 z}{dy^2}\delta y^2 = -\frac{y}{z}\delta y - \frac{y^2+z^2}{2z^3}\delta y^2,$$

which will at once appear if we remember that $x$ has no variation, and that $\delta y$ and $\delta z$ are taken along any section of the sphere at right angles to the axis of $x$. Substituting this value of $\delta z$, we shall evidently obtain, as the coefficient of $\delta y$, $M'$, as before, which must be equated to zero as formerly. But since we must not reject the new term of the second order arising

from the elimination of $\delta z$, we add it to those already in the second, and the complete terms then become

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\left\{S + \frac{d}{dx}\frac{z'}{\sqrt{1+y'^2+z'^2}}\cdot\frac{y^2+z^2}{z^2}\delta y^2\right\}dx.$$

When the limiting values of $x, y$ and $z$ are fixed, and the arc joining the two fixed points is less than a semi-circumference, the sign of these terms is undoubtedly positive, as we know from other considerations that we have a minimum. Nevertheless the author is unable to present any satisfactory general demonstration of the fact that these terms fulfil all the necessary conditions for a minimum.

**320.** The method employed in the preceding problem is not sufficiently general for all cases, since it is evident that the connecting equation may, as when $s$ is the independent variable, be an unintegrable differential equation, which will not enable us to express $\delta z$ in terms involving $\delta y$ only. In this case we must adopt the method of Lagrange, with which the reader is already partially familiar, and which we will now briefly explain in a somewhat more general manner.

**321.** Suppose we seek to maximize or minimize the expression $U = \int_{x_0}^{x_1} V dx$, where $V$ is any function of $x$, $y$, $y'$, .... $z$, $z'$, ....; and suppose also that the equation $f(x, y, z, y', z', \ldots) = 0 = f$ is always to hold. Then, because $f$ is always zero, $\delta f$ must vanish; that is, we must have

$$f_y\delta y + f_{y'}\delta y' + \text{etc.} + f_z\delta z + f_{z'}\delta z' + \text{etc.} = 0. \qquad (1)$$

Then $l$ being any quantity, constant or variable, we may write $\int_{x_0}^{x_1} l\delta f\, dx = 0$. Now vary $U$ to the first order, equate $\delta U$ to zero, and transform by integration as usual. Then, in like

manner, transform $\int_{x_0}^{x_1} l\delta f\,dx$, and add the result to $\delta U$. Then giving only the general form of the terms free from the integral sign, we shall have

$$\begin{aligned}\delta U = &\{V_{y'} + lf_{y'} - (V_{y''} + lf_{y''})' + \text{etc.}\}\delta y + \text{etc.}\\ &+\{V_{z'} + lf_{z'} - (V_{z''} + lf_{z''})' + \text{etc.}\}\delta z + \text{etc.}\\ &+\int_{x_0}^{x_1}\{[V_y + lf_y - (V_{y'} + lf_{y'})' + \text{etc.}]\,\delta y\\ &\qquad + [V_z + lf_z - (V_{z'} + lf_{z'})' + \text{etc.}]\,\delta z\}dx = 0. \qquad (2)\end{aligned}$$

Now whatever be the value of $l$, the integrated and the unintegrated parts of $\delta U$ must severally vanish. Then if we assume $l$ so as to make the coefficients of either of the quantities $\delta y\,dx$ or $\delta z\,dx$ vanish, the coefficient of the other must vanish also. Thus we reduce $\delta U$ to such a form that, without eliminating either $\delta y$ or $\delta z$, we may, in the unintegrated part, regard these quantities as if they were really independent, and equate their coefficients severally to zero. But it is evident that before we can obtain $y$ and $z$ as functions of $x$, we must be able either to eliminate $l$ or to determine it also as a function of $x$, $y$ and $z$. This, however, can in general be accomplished, because, in addition to the differential equations obtained by equating to zero the coefficients of $\delta y$ and $\delta z$, we now have the equation $f = 0$.

**322.** The arbitrary constants which enter the general solution must evidently be determined by the conditions which are to hold at the limits. Denoting the terms at the limits by $L_1 - L_0$, we may evidently in general equate these quantities severally to zero. Then in $L_1$, for example, the value which we have been obliged to assign to $l$ will not usually cause the coefficients of $dx_1$, $\delta y_1$, $\delta y_1'$, $\delta z_1$, $\delta z_1'$, etc., to vanish severally. Moreover, these variations are not independent, because the

equation $f = 0$ is to hold among them; so that we cannot equate these coefficients severally to zero.

We see, therefore, that although we shall have as many equations at the limits as there are independent variations, a general discussion of the number of the arbitrary constants involved in the general solution, and of the number of the ancillary equations for their determination, must become complicated. Some results relative to these points have been obtained by Prof. Jellett, and are given in Art. 59 of his work, which results appear to be correct, although, as he himself states, they are at variance with some obtained by Poisson. It will here be sufficient to present Prof. Jellett's conclusions without demonstration.

Let $V$ be of the order $n$ in $y$ and $m$ in $z$, and let the connecting equation $f = 0$ be of the order $n'$ in $y$ and $m'$ in $z$, the limiting values of $x$ only being fixed.

Then, first supposing $n > n'$ and $m > m'$, the number of constants involved in the general solution will be the greater of the two quantities $2(m + n')$ and $2(m' + n)$, while the number of the independent variations remaining in $L_1 - L_0$, whose coefficients may be equated to zero, will be the same; so that all the constants can in this case be determined; and the same conclusion holds when $m > m'$ and $n < n'$.

If we next suppose $n < n'$ and $m < m'$, the number of constants involved in the general solution will in general be $2(m' + n')$, and of these constants there may remain undetermined any number not exceeding the lesser of the two quantities $2(m' - m)$ and $2(n' - n)$.

**323.** The method of Lagrange is capable of extension to any number of dependent variables. For example, let $V$ contain $x, y, z, u$, and any differential coefficients of $y, z$ and $u$ with respect to $x$; and let the equations

$$f(x, y, z, u, y', z', u', \text{etc.}) = 0 = f$$

and

$$F(x, y, z, u, y', z', u', \text{etc.}) = 0 = F$$

always hold. Then both $\delta f$ and $\delta F$ must vanish; and assuming $l_{,}$ as another undetermined quantity, we have

$$\int_{x_0}^{x_1} l\,\delta f\,dx = 0 \quad \text{and} \quad \int_{x_0}^{x_1} l_{,}\delta F\,dx = 0.$$

Adding both these equations to $\delta U$, and transforming as usual, the integrated and the unintegrated parts must severally vanish. Then we may write

$$\int_{x_0}^{x_1} (M\delta y + N\delta z + P\delta u)\,dx = 0;$$

and if we so assume $l$ and $l_{,}$ as to cause any two of the quantities $M$, $N$ and $P$ to vanish, the other must vanish also, after which $l$ and $l_{,}$ must as before be eliminated, or found as a function of $x$, $y$, $z$ and $u$, in order that we may obtain a complete solution.

**324.** If we adopt the method of Lagrange in the preceding problem, we shall obtain the equations

$$ly - \frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2 + z'^2}} = 0, \qquad lz - \frac{d}{dx}\frac{z'}{\sqrt{1 + y'^2 + z'^2}} = 0,$$

so that eliminating $l$ we shall arrive at equation (6), Art. 317. Moreover, $l$ will not, in this case, appear at all in the terms at the limits, which will therefore be of the same form as before.

**325.** The last example is merely an individual of an extensive class, and was discussed separately merely for the sake of introducing in a simple manner the subject of the present section. We shall now proceed to consider a very general problem given by Prof. Jellett, from the partial solution of which the last and many other examples can be readily solved.

## Problem LII.

*Let $v$ be any function of $x$, $y$ and $z$, and let these quantities be also connected by the equation $f(x, y, z) = 0 = f$. Then it is required to maximize or minimize the expression*

$$U = \int_{x_0}^{x_1} v \sqrt{1 + y'^2 + z'^2}\, dx.$$

Supposing $y$ and $z$ found as functions of $x$, we may evidently then regard $x$, $y$ and $z$ as the co-ordinates of some curve; so that we may consider as usual that we require a curve whose co-ordinates shall be the values of $x$, $y$ and $z$ in the general solution, and which we may therefore call the required curve. Moreover, since the equation $f = 0$ may be regarded as the equation of a surface, we may suppose that the curve is required to lie upon this given surface.

Now let $ds$ be an element of this required curve. Then we may, as in the case of two co-ordinates, adopt $s$ as the independent variable, considering $x$, $y$ and $z$ as functions of $s$, which itself receives no variation. We must, however, in this case, adopt the method of Lagrange for three dependent variables. For, since $s$ is to be the independent variable, $x$, $y$ and $z$, besides satisfying the explicit equation $f = 0$, must also satisfy the unintegrable differential equation

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} - 1 = 0 = F.$$

**326.** Now transforming to $s$, we have

$$U = \int_{s_0}^{s_1} v\, ds, \qquad x'^2 + y'^2 + z'^2 = 1, \qquad f = 0, \qquad (1)$$

$$\left.\begin{aligned}\delta U = v_1\,ds_1 - v_0\,ds_0 + \int_{s_0}^{s_1}(v_x\delta x + v_y\delta y + v_z\delta z)\,ds = 0,\\ \int_{s_0}^{s_1} l\,(x'\delta x' + y'\delta y' + z'\delta z')\,ds = 0,\\ \int_{s_0}^{s_1} l_{\prime}(f_x\delta x + f_y\delta y + f_z\delta z)\,ds = 0,\end{aligned}\right\} \quad (2)$$

where accents above denote differentiation with respect to $s$, and literal suffixes partial differentiation as hitherto. Then, proceeding as usual, we obtain

$$\left.\begin{aligned}v_x + l_{\prime}f_x - (lx')' = 0 = v_x + l_{\prime}f_x - lx'' - x'l',\\ v_y + l_{\prime}f_y - (ly')' = 0 = v_y + l_{\prime}f_y - ly'' - y'l',\\ v_z + l_{\prime}f_z - (lz')' = 0 = v_z + l_{\prime}f_z - lz'' - z'l'.\end{aligned}\right\} \quad (3)$$

Multiplying these equations by $x'$, $y'$ and $z'$ respectively, and adding the products employing the equations

$$\left.\begin{aligned}&f_x x' + f_y y' + f_z z' = f' = 0, && x'^2 + y'^2 + z'^2 = 1,\\ &v_x x' + v_y y' + v_z z' = v', && x'x'' + y'y'' + z'z'' = 0,\\ &l_x x' + l_y y' + l_z z' = l',\end{aligned}\right\} \quad (4)$$

we have

$$v' - l' = 0 \qquad \text{and} \qquad l = v + a. \quad (5)$$

**327.** Before proceeding, we must determine the constant $a$, and this will lead us to examine the terms at the limits. These terms are

$$v_1\,ds_1 - v_0\,ds_0 + l_1(x'\delta x + y'\delta y + z'\delta z)_1$$
$$- l_0(x'\delta x + y'\delta y + z'\delta z)_0 = 0. \quad (6)$$

Now if the required curve is to connect two fixed points,

we shall have, by reasoning like that employed in Arts. 276 and 277, for either limit,

$$\delta x = -x'ds, \qquad \delta y = -y'ds, \qquad \text{and} \quad \delta z = -z'ds. \qquad (7)$$

Hence, substituting these values in (6), and observing the second equation (1), we have

$$(v - l)_1 ds_1 - (v - l)_0 ds_0 = 0,$$

so that

$$v_1 = l_1 \qquad \text{and} \quad l = v.$$

But if the required curve is to connect two fixed curves, let the equations for the fixed curve at the upper limit be $dy = p\,dx$ and $dz = q\,dx$. Then, by reasoning precisely like that of Art. 278, we have

$$\left.\begin{aligned} \delta y_1 + y_1'ds_1 &= p_1(\delta x_1 + x_1'ds_1), \\ \delta z_1 + z_1'ds_1 &= q_1(\delta x_1 + x_1'ds_1). \end{aligned}\right\} \qquad (8)$$

Substituting in $L_1$ the values of $\delta y_1$ and $\delta z_1$ found from (8), we have, omitting suffixes,

$$v\,ds + lx'\delta x + ly'(p\,\delta x + px'ds - y'ds)$$

$$+ lz'(q\,\delta x + qx'ds - z'ds) = 0. \qquad (9)$$

But $ds_1$ and $\delta x_1$ are entirely independent, so that, equating their coefficients severally to zero, we have

$$\left.\begin{aligned} v + ly'x'p - ly'^2 + lz'x'q - lz'^2 &= 0, \\ lx' + ly'p + lz'q &= 0. \end{aligned}\right\} \qquad (10)$$

Multiplying the second of these equations by $x'$, and subtracting from the first, observing the second of equations (1), we find, as before, that $l_1 = v_1$ and $l = v$.

Now, supposing $l_1$ or $v_1$ not to vanish, divide the second of equations (10) by $l_1 x_1'$, and we shall obtain

$$1 + \frac{dy_1}{dx_1} p_1 + \frac{dz_1}{dx_1} q_1 = 0,$$

which shows that the required curve must cut the fixed curve at right angles; and a similar result can evidently be obtained for another fixed curve at the lower limit.

**328.** Let us now return to the general solution.

Putting $v$ for $l$ in (3), we have

$$\left.\begin{aligned} v_x + l_, f_x - vx'' - x'v' &= 0, \\ v_y + l_, f_y - vy'' - y'v' &= 0, \\ v_z + l_, f_z - vz'' - z'v' &= 0. \end{aligned}\right\} \qquad (11)$$

Let $A$, $B$ and $C$ be the angles made with the co-ordinate planes by the plane of that normal section which contains at any point the tangent to the required curve. Then, because the plane contains the normal to the surface, and also the tangent to the required curve, we must have the equations

$$\left.\begin{aligned} f_x \cos A + f_y \cos B + f_z \cos C &= 0, \\ x' \cos A + y' \cos B + z' \cos C &= 0. \end{aligned}\right\} \qquad (12)$$

Hence, multiplying the first of equations (11) by $\cos A$, the second by $\cos B$, the third by $\cos C$, and adding the products, we have

$$v_x \cos A + v_y \cos B + v_z \cos C$$
$$- v\,(x'' \cos A + y'' \cos B + z'' \cos C) = 0. \qquad (13)$$

Now let $A_,$, $B_,$ and $C_,$ be the angles which $r_,$, the radius of curvature of the required curve, makes with the co-ordinate axes. Then it is known that

$$\cos A_{,} = -r_{,}x'', \qquad \cos B_{,} = -r_{,}y'', \qquad \cos C_{,} = -r_{,}z''. \quad (14)$$

Equation (13) may therefore be written

$$v_x \cos A + v_y \cos B + v_z \cos C$$

$$= -\frac{v}{r_{,}}(\cos A \cos A_{,} + \cos B \cos B_{,} + \cos C \cos C_{,}). \quad (15)$$

Next let $o$ be the angle which the osculating plane to the curve makes with the plane of the aforesaid normal sections. Then it is known that

$$\cos A \cos A_{,} + \cos B \cos B_{,} + \cos C \cos C_{,} = \sin o. \quad (16)$$

Also, $r_{,,}$ being the radius of curvature of this normal section, we have, by Meunier's Theorem,

$$\sin^2 o = 1 - \frac{r_{,}^2}{r_{,,}^2}. \quad (17)$$

Equation (15) may therefore be written under any one of these three forms:

$$\left.\begin{aligned} \frac{1}{r_{,}^2} - \frac{1}{r_{,,}^2} &= \frac{1}{v^2}(v_x \cos A + v_y \cos B + v_z \cos C)^2, \\ \frac{\sin o}{r_{,}} &= -\frac{1}{v}(v_x \cos A + v_y \cos B + v_z \cos C), \\ \frac{\tan o}{r_{,,}} &= -\frac{1}{v}(v_x \cos A + v_y \cos B + v_z \cos C). \end{aligned}\right\} \quad (18)$$

The first two equations are evident enough, but to obtain the third we have

$$\frac{r_{,}^2}{r_{,,}^2} = \cos^2 o = \frac{\sin^2 o}{\tan^2 o}.$$

Hence $\qquad r_{\prime}^2 : r_{\prime\prime}^2 :: \sin^2 o : \tan^2 o,$

or $\qquad \sin^2 o : r_{\prime}^2 :: \tan^2 o : r_{\prime\prime}^2.$

Whence
$$\frac{\sin^2 o}{r_{\prime}^2} = \frac{\tan^2 o}{r_{\prime\prime}^2}.$$

**329.** Equations (18) are as far as we can carry the general solution, so long as the form of $v$ is altogether undetermined; but we may nevertheless deduce a remarkable property belonging to this class of curves.

Suppose we consider the integral $U_{\prime} = \int_{s_0}^{s_1} \frac{ds}{v} = \int_{s_0}^{s_1} v_{\prime} ds.$ Then, since

$$\frac{dv_{\prime}}{dx} = -\frac{1}{v^2}\frac{dv}{dx}, \qquad \frac{dv_{\prime}}{dy} = -\frac{1}{v^2}\frac{dv}{dy}, \qquad \frac{dv_{\prime}}{dz} = -\frac{1}{v^2}\frac{dv}{dz},$$

the equation in $v_{\prime}$, which will replace (15), will reduce to equation (15) in $v$, except that the sign of the second member will be changed. Therefore the first of equations (18) includes the solution of both problems. Hence, because equation (15) in $v$ can be converted into equation (15) in $v_{\prime}$ by merely changing the sign of $r_{\prime\prime}$, we conclude that if, upon a given surface, a curve whose equation is

$$r_{\prime} = f(x, y, z, y_x)$$

possess the property of rendering $\int_{s_0}^{s_1} v\, ds$ a maximum or a minimum, the curve whose equation is

$$r_{\prime} = -f(x, y, z, y_x)$$

will have the property of rendering $\int_0^{s_1} \frac{ds}{v}$ a maximum or a minimum.

**330.** We may, before considering particular cases, deduce another property of this class of curves.

Let $O$ be a fixed point and $TT'$ a fixed curve, both being situated upon a given surface, and let the arc $TT'$ be taken indefinitely small; then draw two curves $OT$ and $OT'$, each having the property of maximizing or minimizing the expression $U=\int v\,ds$, when the limits are fixed. Then, since each curve renders $U$ a maximum or a minimum, they must both satisfy the same differential equations; and unless we suppose that there could be two solutions, $OT$ and $OT'$ must be indefinitely near at each point, so that $OT'$ could be obtained by varying $OT$, and also the upper limiting value of $x$. Therefore $OT' - OT$ must equal that portion only of $\delta U$ which is without the integral sign—that is, since the lower limit remains fixed, and $l_1 = v_1$—must equal

$$v_1\,ds + v_1 x_1'\delta x_1 + v_1 y_1'\delta y_1 + v_1 z_1'\delta z_1. \tag{19}$$

Denote $TT'$ by $dS$. Then we must have

$$\delta x_1 = \frac{dx_1}{dS_1}dS_1 - \frac{dx_1}{ds_1}ds_1, \qquad \delta y_1 = \frac{dy_1}{dS_1}dS_1 - \frac{dy_1}{ds_1}ds_1,$$

$$\delta z_1 = \frac{dz_1}{dS_1}dS_1 - \frac{dz_1}{ds_1}ds_1.$$

Substituting these values in (19), we have

$$\begin{aligned} OT' - OT &= v_1 - v_1(x'^2 + y'^2 + z'^2)_1\,ds_1 \\ &\quad + v_1\left(x'\frac{dx}{dS} + y'\frac{dy}{dS} + z'\frac{dz}{dS}\right)_1 dS_1 \\ &= v_1\left(x'\frac{dx}{dS} + y'\frac{dy}{dS} + z'\frac{dz}{dS}\right)_1 dS_1. \end{aligned} \tag{20}$$

But denoting by $t$ the angle $OT'T$, (20) gives

$$OT' - OT = v_1 \cos t\, dS. \tag{21}$$

**331.** From the second of equations (18) may be deduced two others, which will sometimes be found useful.

Let $A_{,,}$, $B_{,,}$ and $C_{,,}$ be the angles made by the osculating plane with the co-ordinate planes. Then we have

$$\left.\begin{aligned} \cos A_{,,} = r_{,}(z'y'' - y'z''), \qquad \cos B_{,,} = r_{,}(x'z'' - z'x''), \\ \cos C_{,,} = r_{,}(y'x'' - x'y''). \end{aligned}\right\} \tag{22}$$

Now let the equation of the given surface be

$$dz = z_x dx + z_y dy. \tag{23}$$

Then we have

$$\sin o = \frac{r_{,}\{y'x'' - x'y'' - z_x(z'y'' - y'z'') - z_y(x'z'' - z'x'')\}}{\sqrt{1 + z_x^2 + z_y^2}}. \tag{24}$$

Substituting this value in the second of equations (18), and then eliminating either $x''$ or $y''$ by the equation

$$x'x'' + y'y'' + z'z'' = 0,$$

we have either of the following forms:

$$\left.\begin{aligned} x'' + z_x z'' &= \\ &\frac{\sqrt{1 + z_x^2 + z_y^2}}{v}(v_x \cos A + v_y \cos B + v_z \cos C)y', \\ y'' + z_y z'' &= \\ &- \frac{\sqrt{1 + z_x^2 + z_y^2}}{v}(v_x \cos A + v_y \cos B + v_z \cos C)x'. \end{aligned}\right\} \tag{25}$$

**332.** We may now proceed to consider some particular cases of the foregoing theory.

### Problem LIII.

*It is required to find the line of minimum length which can be drawn upon a given surface between two fixed points or two fixed curves situated upon the same surface.*

This problem is the simplest case of the preceding, to which it may be reduced by writing $v = 1$, $v_x = 0$, $v_y = 0$ and $v_z = 0$. Whence the second members of equations (18) must vanish, and the first of those equations will give $r_{,} = r_{,,}$, which makes $o$ vanish.

We see, therefore, that curves of this class, or, as they are generally termed, *geodesic* or *geodetic* curves, must be such that their osculating plane at any point shall be perpendicular to the given surface at that point.

Now, in Prob. LI., since the radius of every normal section of a sphere is the radius of the sphere itself, it at once appears that every geodesic curve drawn on its surface must be the arc of a great circle.

We shall not, however, enter upon an extended discussion of geodesic lines, but shall give the chief points concerning them, following, as we have done since the beginning of Prob. LII., the guidance mainly of Prof. Jellett.

**333.** The equation of a geodesic line is deducible in several ways.

1st. It is evident that all the reasoning in Prob. LI. would hold if the equation of the given surface were $f = 0$, except that $x$, $y$ and $z$ in the denominators of the equations would be replaced by $f_x$, $f_y$ and $f_z$ respectively, the equations remaining otherwise unaltered. This would hold as far as equation (9), which would become

$$\frac{x''}{f_x} = \frac{y''}{f_y} = \frac{z''}{f_z}.$$

Hence we may evidently write the equations

$$\left.\begin{array}{c} f_x d^2y - f_y d^2x = 0, \qquad f_x d^2z - f_z d^2x = 0, \\ f_y d^2z - f_z d^2y = 0. \end{array}\right\} \tag{1}$$

2nd. If the equation of the given surface be

$$dz = z_x dx + z_y dy,$$

then equations (25) give

$$x'' + z_x z'' = 0, \qquad y'' + z_y z'' = 0. \tag{2}$$

3rd. Or, because $o = 0$, we have, from (24),

$$z_x (y'z'' - z'y'') + z_y (z'x'' - x'z'') = x'y'' - y'x''. \tag{3}$$

But we have the equations

$$dz = z_x dx + z_y dy,$$

$$d^2z = z_{xx} dx^2 + 2z_{xy} dx\, dy + z_{yy} dy^2 + z_y d^2y.$$

Eliminating $dz$ and $d^2z$ in (3) by the values just given, it becomes

$$(1 + z_x^2 + z_y^2) y_{xx} + (z_y - z_x y_x)(z_{xx} + 2z_{xy} y_x + z_{yy} y_{xx}) = 0,$$

which, together with the equation $dz - z_x dx - z_y dy = 0$, represent the geodesic line.

**334.** As another property connected with these lines, we may notice that equation (21), Art. 330, will now become $OT' - OT = \cos t\, dS$. Now it is easily shown by the differential calculus that if the lines $OT$ and $OT'$ were right lines, the point $O$ and the curve $TT'$ being in free space, the above formula would hold. Therefore we infer that, so far as their

change of length is concerned, we may, in the application of infinitesimals, regard all geodesic lines as right lines.

But we must here note a difference between the right line and most other geodesic lines. For while the former is always the line of minimum length between two fixed points, the latter are not in every case. For on a given surface suppose two indefinitely near geodesic lines to be drawn: these lines will in general intersect at some point. Here, then, we would have between two points two indefinitely near geodesic curves satisfying the same differential equations, so that we would naturally infer, as in the case of two co-ordinates only, that the geodesic line ceases to be a minimum when the integral $U$ ranges from one intersection to the other. This remark, although undoubtedly true, can only be called an inference, as we cannot apply Jacobi's Theorem with any success to this class of problems; still the principle in question is well illustrated in the case of the sphere, where the two geodesic curves will be two indefinitely near semi-circumferences, having of course the same length.

Again, we have already seen that the curve which renders $\int_{s_0}^{s_1} v\,ds$ a maximum or a minimum must, if one or both extremities are to be confined to fixed curves, cut its limiting curve at right angles, and hence this must be true also of geodesic curves. If, therefore, from a fixed point upon a given surface geodesic curves of a given length be drawn in every direction, and the extremities, which are free, be joined by a curve, this curve must be of such a nature that at any point it may be at right angles to the geodesic curve drawn from the common point to that point.

**335.** We close the discussion of this subject by the consideration of one more particular geodesic curve, the discussion of which is not without interest, and appears to be due to Joachimstal.

## Problem LIV.

*It is required to determine the nature of the geodesic curve drawn upon the surface of a spheroid.*

Let the equation of the surface be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1. \tag{1}$$

Hence equations (2), Art. 333, become

$$x''=\frac{b^2xz''}{a^2z}, \qquad y''=\frac{yz''}{z}. \tag{2}$$

Now let $P$ denote the perpendicular from the centre upon the tangent plane to the surface at any point of the required curve, and $D$ the semi-diameter of the spheroid drawn parallel to the tangent of the required curve through the same point. Then it is known that

$$\left.\begin{aligned}\frac{1}{P^2}&=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{b^4}=u\\ \text{and}\qquad \frac{1}{D^2}&=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{b^2}=v.\end{aligned}\right\} \tag{3}$$

Now differentiating $v$, and putting for $x''$ and $y''$ their values from (2), we have

$$v'=\frac{2b^2}{z}\left(\frac{xx'}{a^4}+\frac{yy'}{b^4}+\frac{zz'}{b^4}\right)z''=\frac{b^2u'z''}{z}. \tag{4}$$

But if we differentiate (1) twice, and in the result substitute for $x''$ and $y''$ their values from (2), observing also the values of $u$ and $v$, we shall obtain

$$v=-\frac{b^2uz''}{z}. \tag{5}$$

Dividing (4) by (5), we have

$$\frac{v'}{v} = -\frac{u'}{u}, \qquad udv + vdu = 0, \tag{6}$$

whence $uv$ is a constant, say $c$.

It appears, therefore, from the first and third members of equations (3) that the geodesic curve must be of such a nature as will always render $PD$ a constant, say $c'^2$.

**336.** We can also deduce another property of this curve. Let $r$ be the radius of curvature. Then it is well known that for any curve in space we have

$$\frac{1}{r^2} = x''^2 + y''^2 + z''^2. \tag{7}$$

Substituting in this equation for $x''^2$ and $y''^2$ their values from (2), and observing also the value of $u$, we have

$$\frac{1}{r^2} = \frac{uz''^2 b^4}{z^2}. \tag{8}$$

But from (5) we have

$$\frac{b^4 z''^2}{z^2} = \frac{v^2}{u^2} = \frac{P^4}{D^4},$$

so that

$$\frac{1}{r^2} = \frac{v^2}{u} = \frac{P^2}{D^4}, \qquad r = \frac{D^2}{P}. \tag{9}$$

But we have $PD = c'^2$ and $D^2 = \frac{c'^4}{P^2}$; and therefore (9) gives $r = \frac{c'^4}{P^3}$.

We see, then, that the radius of curvature for the geodesic curve must vary inversely as the cube of the perpendicular

drawn from the centre of the spheroid to its tangent plane which touches the geodesic curve.

**337.** We now pass to another problem which is of considerable interest, following, as before, the guidance mainly of Prof. Jellett.

## Problem LV.

*A particle is compelled to move in a groove upon a given surface from one fixed point to another, being urged by a system of forces which always render $X\,dx + Y\,dy + Z\,dz$ a perfect differential. Then it is required to determine the nature of its path in order that it may move from the first point to the second in a minimum time.*

Let $t$ denote the time, $V$ the tangential velocity, and $ds$ an element of the required path. Then it is evident that we are to determine the curve which will minimize the expression

$$U = \int_{s_0}^{s_1} \frac{ds}{V}.$$

But we have, by a well-known principle of mechanics,

$$V^2 = 2\int (X\,dx + Y\,dy + Z\,dz) = f(x, y, z).$$

Hence we conclude that the present problem is merely another case of Prob. LII., to which it may be reduced by writing $\frac{1}{V} = v$.

But before we can employ any of the formulæ obtained in that problem, we must also be able to determine the values of the partial differential coefficients of $v$ with respect to $x$, $y$ and $z$. Now we have

$$\frac{1}{2}\,d(V^2) = V\,dV = X\,dx + Y\,dy + Z\,dz, \qquad (1)$$

where the differentiation is total. But since $V$, and consequently $V^2$, is a function of $x, y$ and $z$, the partial differential of $V^2$ with respect to any of these variables, as $x$, is the change which it would undergo if we could change $x$ into $x + dx$, the other variables remaining unaltered, and this change is given in (1) by making $dy$ and $dz$ zero. Hence, denoting partial differential coefficients, as before, by literal suffixes, we have

$$\left.\begin{aligned} \frac{1}{2}(V^2)_x = VV_x = X, \\ VV_y = Y, \qquad VV_z = Z. \end{aligned}\right\} \tag{2}$$

Now putting for $v$ its value $\frac{1}{V}$, we find

$$\left.\begin{aligned} v_x = -\frac{V_x}{V^2} = -\frac{VV_x}{V^3} = -\frac{X}{V^3}, \\ v_y = -\frac{Y}{V^3}, \qquad v_z = -\frac{Z}{V^3}. \end{aligned}\right\} \tag{3}$$

Substituting these values together with that of $V$ in the first of equations (18), Art. 328, we obtain

$$\frac{1}{r_{\prime}^2} - \frac{1}{r_{\prime\prime}^2} = \frac{1}{V^4}(X\cos A + Y\cos B + Z\cos C)^2. \tag{4}$$

**338.** Although we cannot carry the solution any further while the problem retains its present general form, yet we can deduce some interesting properties belonging to this class of brachistochrone curves.

Let $R$ denote the resultant of the forces at any point of the required curve, $o$ the angle made by the osculating plane to the curve at that point with the plane of that normal section which contains the tangent to the required curve at that

point, and $O$ the angle which $R$ makes with the perpendicular to the plane of this normal section erected at the aforesaid point. Then we know that

$$\frac{r_{\prime}^2}{r_{\prime\prime}^2} = \cos^2 o, \qquad 1 - \frac{r_{\prime}^2}{r_{\prime\prime}^2} = \sin^2 o, \qquad \frac{1}{r_{\prime}^2} - \frac{1}{r_{\prime\prime}^2} = \frac{\sin^2 o}{r_{\prime}^2}. \qquad (5)$$

Now the aforesaid perpendicular to the plane of the normal section makes angles with the co-ordinate planes whose cosines are numerically equal to $\cos A$, $\cos B$ and $\cos C$. Hence we see that

$$(X\cos A + Y\cos B + Z\cos C)^2 = R^2\cos^2 O. \qquad (6)$$

Therefore we have

$$\left.\begin{array}{c} \dfrac{\sin^2 o}{r_{\prime}^2} = \dfrac{R^2\cos^2 O}{V^4}, \qquad \dfrac{V^4\sin^2 o}{r_{\prime}^2} = R^2\cos^2 O, \\[2ex] \dfrac{V^2\sin o}{r_{\prime}} = R\cos O. \end{array}\right\} \qquad (7)$$

**339.** It is evident that the members of the last equation may, so far as the preceding equations are concerned, have contrary signs, and we must therefore next justify our assumption that they should be taken alike.

Now the pressure upon the curve in any direction is equal to the sum of the components, in that direction, of the resultant and of the centrifugal force. Moreover, the total force at any point may be resolved into three: the first normal to the surface, which is destroyed by the surface; the second along the tangent to the required curve, which tends to produce acceleration of motion; and the third in the direction of that perpendicular which has been previously mentioned, and this component would, if the particle were constrained to move in a groove, cause pressure against its side. But $\dfrac{V^2}{r'}$ is the cen-

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$X \cos A + Y \cos B + Z \cos C = 0,$$

which, in (4), gives

$$\frac{1}{r_{\prime}^{2}} - \frac{1}{r_{\prime\prime}^{2}} = 0.$$

Therefore the curve in this case must be a geodesic curve.

**341.** The following problem is also from the work of Prof. Jellett, and its complete solution appears to be due to him, although the problem itself had been previously discussed by Delaunay.

### Problem LVI.

*It is required to determine the nature of the curve of minimum length which can be drawn between two fixed points in free space, the radius of curvature of the curve being always an assigned constant.*

Let $ds$ be an element of the required curve, and $r$ the radius of curvature, which is a constant. Then, adopting here also the arc as the independent variable, we are to determine the curve which will minimize the expression $U = \int_{s_0}^{s_1} ds$. We have also, in order that we may be able to employ the method of Lagrange, the two additional equations

$$x'^2 + y'^2 + z'^2 = 1, \qquad x''^2 + y''^2 + z''^2 = \frac{1}{r^2} = R^2. \qquad (1)$$

Therefore, since the variation of $U$ can give only the terms $ds_1 - ds_0$, it is easy to see that by following the method of Lagrange, $l$ and $l_{\prime}$ being two undetermined quantities, we shall obtain the equations

$$\left.\begin{matrix} (l_{\prime}x'')'' - (lx')' = 0, \qquad (l_{\prime}y'')'' - (ly')' = 0, \\ (l_{\prime}z'')'' - (lz')' = 0. \end{matrix}\right\} \qquad (2)$$

Whence, by integration,

$$\left.\begin{aligned}(l_{,}x'')' - lx' &= a = x''l_{,}' + l_{,}x''' - lx',\\ (l_{,}y'')' - ly' &= b = y''l_{,}' + l_{,}y''' - ly',\\ (l_{,}z'')' - lz' &= c = z''l_{,}' + l_{,}z''' - lz'.\end{aligned}\right\} \qquad (3)$$

Eliminating $l$ between the first and second of these equations, we obtain

$$(y'x'' - x'y'')\, l_{,}' + (y'x''' - x'y''')\, l_{,} = ay' - bx'.$$

This equation is immediately integrable, giving

$$l_{,}\,(y'x'' - x'y'') = ay - bx + f. \qquad (4)$$

In like manner, eliminating $l$ between the third and first, and then between the second and third of these equations, and integrating the two resulting equations, we have

$$\left.\begin{aligned}l_{,}\,(x'z'' - z'x'') &= cx - az + f_{,},\\ l_{,}\,(z'y'' - y'z'') &= bz - cy + f_{,,}.\end{aligned}\right\} \qquad (5)$$

**342.** Before proceeding further we must consider the mode of determining the constants in (4) and (5), and we begin by determining $l_{,}$ and $l$. For this purpose, multiply the first of equations (3) by $x''$, the second by $y''$, the third by $z''$, and add the products, observing that equations (1) hold, and that hence

$$x'x'' + y'y'' + z'z'' = 0 \quad \text{and} \quad x''x''' + y''y''' + z''z''' = 0. \qquad (6)$$

Then we have

$$R^2 l_{,}' = ax'' + by'' + cz'', \qquad R^2 l_{,} = ax' + by' + cz' + g. \qquad (7)$$

Differentiating the first of equations (6), and transposing, we have

$$x'x''' + y'y''' + z'z''' = -(x''^2 + y''^2 + z''^2) = -R^2. \qquad (8)$$

Now multiply the first of equations (3) by $x'$, the second by $y'$, the third by $z'$, and add the products, observing the first of equations (1), and also equation (8). Then we obtain

$$l = -R^2 l_{,} - ax' - by' - cz'. \tag{9}$$

Hence, by the second of equations (7), we have

$$l = g - 2R^2 l_{,}. \tag{10}$$

**343.** We must next consider the terms at the limits. Giving merely their general form, these are:

$$L = ds + \{lx' - (l_{,}x'')'\}\delta x + \{ly' - (l_{,}y'')'\}\delta y + \{lz' - (l_{,}z'')'\}\delta z$$
$$+ l_{,}\{x''\delta x' + y''\delta y' + z''\delta z'\} = 0. \tag{11}$$

Now suppose the extremities of the required curve to be fixed, but the extreme tangents to be wholly unrestricted. Then it is evident, first of all, that $L_1$ and $L_0$ must severally vanish.

Now consider $L_1$, and take first those terms only which contain $ds_1$, $\delta x_1$, $\delta y_1$ and $\delta z_1$. Then, because the extreme points are fixed, we shall have, as usual,

$$\delta x_1 = -x_1'ds_1, \qquad \delta y_1 = -y_1'ds_1, \qquad \delta z_1 = -z_1'ds_1, \tag{12}$$

which being substituted in that part of $L_1$, having first written

$$(l_{,}x'')' = l_{,}'x'' + l_{,}x''', \text{ etc.},$$

will give, by employing the first of equations (1) and (6), and also equation (8),

$$(1 - l + R^2 l_{,})_1 ds_1. \tag{13}$$

Now it is evident that we could, without restricting the extreme tangents, so vary the arc as to produce no change in its length, in which case $ds_1$ would vanish, and the remaining

part of $L_1$ would then vanish also. Hence we see that the two parts of $L_1$ are independent, and we have

$$L_{\prime 1}(x''\delta x' + y''\delta y' + z''\delta z')_1 = 0. \tag{14}$$

As this equation can be satisfied by making either factor zero, let us suppose the second to vanish. Then, although $\delta x_1'$, $\delta y_1'$ and $\delta z_1'$ are not independent, we have, from the first of equations (1),

$$x'\delta x' + y'\delta y' + z'\delta z' = 0\,; \tag{15}$$

and if, by this equation, we eliminate any one of the variations, as $\delta z_1'$, the two remaining variations may be regarded as independent, and their coefficients be equated severally to zero.

Now in the second factor of (14) first eliminate $\delta z_1'$, and equate to zero the coefficients of $\delta x_1'$ and $\delta y_1'$; then eliminate $\delta y_1'$, and equate to zero those of $\delta x_1'$ and $\delta z_1'$. Then we shall obtain

$$(z'x'' - x'z'')_1 = 0, \quad (z'y'' - y'z'')_1 = 0, \quad (x'y'' - y'x'')_1 = 0. \tag{16}$$

If now we square these equations and add them, and then to the sum add the square of the first of equations (6), we shall obtain a result which may be written

$$(x''^2 + y''^2 + z''^2)_1\,(x'^2 + y'^2 + z'^2)_1 = R^2, \tag{17}$$

the last member following from equations (1). This would make the radius of curvature infinite at the upper limit; and as it is to have a constant value throughout $U$, the required curve would become a right line. But if we reject this solution and require that the radius of curvature shall have a constant finite value, the second factor of (14) cannot vanish, and $l_{\prime 1}$ must vanish.

Now since the coefficient of $ds_1$ in (13) must also vanish, we see that $l_1$ must become equal to unity. These values make $g$

in equation (10) also unity, and the second of equations (7) becomes

$$l_{,}R^{2} = ax' + by' + cz' + 1. \tag{18}$$

**344.** It is evident that we can treat $L_0$ in a similar manner, and shall obtain like results; so that we have

$$l_{,1} = 0, \qquad l_{,0} = 0, \qquad l_1 = 1, \qquad l_0 = 1. \tag{19}$$

Now since the position of the origin is in our power, assume it at the lower limit. Then, since $x_0$, $y_0$ and $z_0$ must vanish, we see at once from equations (4) and (5) that $f$, $f_{,}$ and $f_{,,}$ must severally vanish. Then, neglecting $f$, $f_{,}$ and $f_{,,}$, multiply equations (4) and (5) by $z'$, $y'$ and $x'$ respectively, and add the products. Then we shall find

$$(ay - bx)z' + (cx - az)y' + (bz - cy)x' = 0$$

$$= a(yz' - zy') + b(zx' - xz') + c(xy' - yx'). \tag{20}$$

To integrate this equation, put $uv$ for $y$, and $xv$ for $z$. Then (20) becomes

$$\frac{du}{au - b} = \frac{dv}{av - c},$$

so that, by integration,

$$l(au - b) = l(av - c) + c_{,} = l(av - c) + lc_{,,} = lc_{,,}(av - c).$$

Now putting for $u$ and $v$ their values, removing the logarithmic sign, and clearing fractions, we have

$$ay - bx = c''(az - cx), \tag{21}$$

which, being an equation of the first degree between three variables, is the equation of a plane, and, containing no constant term, the plane passes through the origin. Now the

circle is the only plane curve of constant finite curvature, and this must therefore constitute the solution required.

**345.** But it is easy to see that the solution just obtained cannot always be applicable. For suppose the assigned value of $r$ to be less than one half the line $AB$, $A$ and $B$ being the two fixed points. Then the circle whose radius is $r$ cannot pass through both points, so that we are led to expect that if there can be any solution in such a case, it must be discontinuous.

Now as no boundary presents itself along which the variations of $x$, $y$ and $z$ are subject to any other restrictions than those which are imposed by equations (1), we infer that the discontinuous solution can consist only of some combination of arcs which satisfy equations (2), and consequently equations (3), which may be regarded as fundamental. Still it is evident that we may, as usual in cases of discontinuity, suppose $a$, $b$ and $c$ to have each different values for the different points of the discontinuous solution.

But in the present case these constants cannot change their values. For let $x_2$, $y_2$ and $z_2$ be the co-ordinates of the point in which two of the arcs which make up the discontinuous solution meet. Then the part of $\delta U$ without the integral sign corresponding to this point, considered as being on the first arc, will involve only $\delta x_2$, $\delta y_2$, $\delta z_2$, $\delta x_2'$, $\delta y_2'$ and $\delta z_2'$. For it is only necessary to add the increment $ds$ to the extreme limits $s_0$ and $s_1$, as the only reason why such increments are required is that we may obtain the privilege of varying the arc in the most general manner, which would require an increase or decrease in its length as a whole. Now the coefficients of $\delta x_2$, $\delta y_2$ and $\delta z_2$ are the first members respectively of equations (3), with contrary signs; so that, denoting this part of $\delta U$ by $L_2$, we have

$$L_2 = -a\delta x_2 - b\delta y_2 - c\delta z_2 + l_{,2}(x''\delta x' + y''\delta y' + z''\delta z')_2. \quad (22)$$

If now we denote by $x_3$, $y_3$ and $z_3$ the co-ordinates of the same points considered as being upon the second arc, we shall have at that point, as in the case of two co-ordinates, the terms $L_2 - L_3$, $L_3$ having the same form as $L_2$. Hence these terms will not vanish unless $a$, $b$ and $c$ have the same values for each arc.

**346.** It appears, then, that the solution must consist of some combination of circular arcs, all having the radius $r$, and situated in the same plane. But $l_{,}$ must vanish at the extreme limits; and we see from (22) that to make $L_2 - L_3$ vanish, we must also make $l_{,2}$ and $l_{,3}$ severally vanish, or must have

$$x_2' = x_3', \qquad y_2' = y_3', \qquad z_2' = z_3', \qquad l_{,2} = l_{,3}. \tag{23}$$

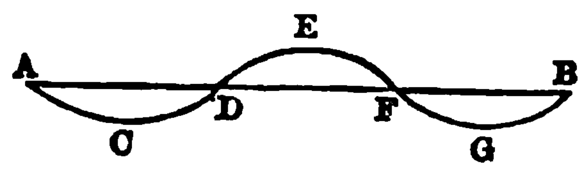
We see, also, from (18), that when the first three of equations (23) are satisfied, the last will be satisfied also; so that we infer that the arcs are also to be so placed as to have a common tangent at their point of meeting, unless, indeed, we can make $l_{,2}$ and $l_{,3}$ vanish without such a construction.

Now since $a$, $b$ and $c$ are unchangeable throughout the integral, (4) and (5), which are derived from the fundamental equations (3), must also hold, as must equation (18); and as the arcs must lie in one plane, we need no longer employ three co-ordinates. Assuming, therefore, the plane of the arcs as that of $xy$, make $z$ and its differentials zero. Then, because $l_{,}$ must vanish at both extreme limits, while $x_0 = 0$ and $y_0 = 0$, it is clear that $f$, $f_{,}$, $f_{,,}$ and $c$ must vanish, so that equations (4), (5) and (18) become respectively

$$l_{,}(y'x'' - x'y'') = ay - bx, \qquad l_{,} = r^2(1 + ax' + by'). \tag{24}$$

**347.** It appears, then, that arcs of the same circle, so joined as to have a common tangent, will give at least one solution of the problem, provided equations (24) are satisfied throughout the entire range of the integration; and this point we next proceed to consider.

Let $A$ and $B$ be the two fixed points, and suppose we take three arcs, $ACD$, $DEF$ and $FGB$; and moreover, since the origin only is fixed, it being at $A$, let the axis of $x$ take the direction $AB$.



Now taking first the arc $ACD$, its general equation may be written

$$(x-h)^2+(y-k)^2=r^2; \tag{25}$$

where $h$ and $k$ are the co-ordinates of the centre; and we shall suppose $x$ and $y$ to be so estimated as to render these co-ordinates positive. Differentiating (25), we have

$$(x-h)x'+(y-k)y'=0. \tag{26}$$

Substituting from (26) in the first of equations (1), we easily find

$$x'=\pm(y-k)R, \qquad y'=\mp(x-h)R. \tag{27}$$

Now if we suppose $x$ and $s$ to increase together, $x'$ is always positive; and $y-k$ being negative, we must take the negative sign. But the arc being below $x$, $y'$ will be positive or negative according as $x-h$ is positive or negative; so that for it we take the positive sign.

We therefore have for this arc

$$\left.\begin{array}{ll} x'=-R(y-k), & y'=R(x-h), \\ x''=-Ry'=-R^2(x-h), & y''=Rx'=-R^2(y-k). \end{array}\right\} \tag{28}$$

Substituting these values in equations (24), they become

$$-Rl_{,}=ay-bx, \qquad l_{,}=r^2\{1+a(y-k)R+b(x-h)R\}. \tag{29}$$

Substituting in the first of these equations the value of $l_{,}$ from the second, we obtain

$$r + ak - bh = 0. \tag{30}$$

Now consider the arc $DEF$, and let $H$ and $K$ be the coordinates of its centre. Then, proceeding as before, we shall find that we must now reverse the signs of $x'$ and $y'$, which will leave those of $x''$ and $y''$ unchanged, and equations (24) will become

$$Rl_{,} = ay - bx, \qquad l_{,} = r^2\{1 + a(y - K)R - b(x - H)R\}.$$

Whence we obtain, as before,

$$r - aK + bH = 0. \tag{31}$$

But since the arcs have the same radius, and a common tangent at $D$, we must have $K = -k$; so that (30) and (31) give $b(h + H) = 0$, an impossible equation unless $b$ vanish. Under this supposition, the second of equations (24) becomes

$$l_{,} = r^2(1 + ax'). \tag{32}$$

But $x'$, being always positive, has the same value at $D$ as at $H$, and therefore, since $l_{,}$ must vanish at the latter point, it will vanish at the former also.

If, on the other hand, we had required for the arc $ACD$ the conditions which would cause $l_{,}$ to vanish at $D$, as well as at $A$, we would have found it necessary to make $b$ vanish, because, while the value of $x'$ is the same at both points, those of $y'$ are numerically equal, but have contrary signs, and therefore the second of equations (24) could not otherwise be satisfied. Then equations (30) and (31) would become $r + ak = 0$ and $r - aK = 0$, so that $K = -k$, as before.

It appears, moreover, from (32), that if $l_{,}$ vanish at $A$ and $D$, it will also vanish at $F$; and that if we had taken any num-

ber of arcs, instead of three, $l_{,}$ would vanish at each point of junction.

**348.** We see, then, that the proposed system of arcs not only gives a solution which satisfies equations (23), but it is also that which is necessary in order that $l_{,}$ may vanish at each point at which discontinuity occurs, so that we have no reason to expect any other solution.

But as we may take as many arcs as we please, all having the assigned radius, it is evident that we can make the system differ practically in no respect from a right line, which was a former solution.

**349.** We have thus far supposed that the curve is to be drawn between two fixed points, but let us next require its extremities to be confined to two surfaces whose equations are $v = 0$ and $V = 0$, $v$ and $V$ being functions of $x$, $y$ and $z$ only. Then, considering the upper limit, we see that $L_1$ becomes, by the aid of (3),

$$L_1 = ds_1 - a\delta x_1 - b\delta y_1 - c\delta z_1 + l_{,1}(x''\delta x' + y''\delta y' + z''\delta z')_1 = 0; \quad (33)$$

and since $l_{,1}$ vanishes, we have

$$L_1 = ds_1 - a\delta x_1 - b\delta y_1 - c\delta z_1 = 0. \quad (34)$$

Now let $x_1 + [\delta x_1]$, $y_1 + [\delta y_1]$ and $z_1 + [\delta z_1]$ be the coordinates of the point in which the required curve, after having been varied, meets the surface. Then we have, omitting the suffix 1,

$$v_x[\delta x] + v_y[\delta y] + v_z[\delta z] = 0. \quad (35)$$

But we have in this case

$$[\delta x] = \delta x + x'ds, \qquad [\delta y] = \delta y + y'ds, \qquad [\delta z] = \delta z + z'ds;$$

so that (35) becomes

$$(v_x\delta x + v_y\delta y + v_z\delta z)_1 + (v_x x' + v_y y' + v_z z')_1 ds_1 = 0. \quad (36)$$

Substituting the value of $ds_1$ from (34), we obtain

$$\{v_x + a(v_x x' + v_y y' + v_z z')\}_1 \delta x_1$$
$$+ \{v_y + b(v_x x' + v_y y' + v_z z')\}_1 \delta y_1$$
$$+ \{v_z + c(v_x x' + v_y y' + v_z z')\}_1 \delta z_1 = 0. \quad (37)$$

We may now regard $\delta x_1$, $\delta y_1$ and $\delta z_1$ as independent, and may therefore equate their coefficients severally to zero. Performing this operation, we easily deduce

$$\frac{v_x}{a} = \frac{v_y}{b} = \frac{v_z}{c}; \quad (38)$$

and a similar equation in $V$ evidently holds for the lower limit. Now from equations (4) and (5), since $l_{\prime}$ vanishes at both limits, and $f$, $f_{\prime}$ and $f_{\prime\prime}$ are zero, we have

$$\left.\begin{aligned} ay_1 - bx_1 &= 0, & ay_0 - bx_0 &= 0, \\ cx_1 - az_1 &= 0, & cx_0 - az_0 &= 0, \\ bz_1 - cy_1 &= 0, & bz_0 - cy_0 &= 0. \end{aligned}\right\} \quad (39)$$

Whence, by subtraction, we deduce

$$\frac{x_1 - x_0}{a} = \frac{y_1 - y_0}{b} = \frac{z_1 - z_0}{c}. \quad (40)$$

Therefore, from (38), we obtain

$$\left(\frac{v_x}{x_1 - x_0}\right)_1 = \left(\frac{v_y}{y_1 - y_0}\right)_1 = \left(\frac{v_z}{z_1 - z_0}\right)_1, \quad (41)$$

and a similar equation in $V$ for the lower limit.

These equations show that the straight line joining the extremities of the arc must be normal to the two given surfaces.

**350.** We have hitherto supposed the extreme tangents of the required curve to be wholly unrestricted; but if we require these tangents to have certain assigned directions, it is evident that the preceding figure cannot always give the general solution of the problem, since these tangents might be so assigned as not to lie in the same plane.

It is shown in the following manner by Prof. Todhunter, in his History of Variations, Art. 156, that the solution in such cases will sometimes be a helix. The discontinuous solution will be found in Art. 154.

Since $\delta x'$, $\delta y'$ and $\delta z'$ are zero at both limits, it is no longer certain that $l_{,}$ will vanish at either limit. Let us suppose, however, that the conditions relative to the limits are such that in equations (4) and (5), $a$, $b$, $f_{,}$ and $f_{,,}$ vanish. Then the second of equations (7) becomes

$$R^2 l_{,} = cz' + g. \tag{42}$$

Also, the terms at the upper limit will now become

$$L_1 = ds_1 - c\delta z_1 = 0;$$

and the extremities of the curve being fixed, $\delta z_1 = -z_1'ds_1$; so that we have

$$1 + cz_1' = 0. \tag{43}$$

But $L_1$ also gives rise to equation (13), so that

$$R^2(l_{,1} - l_1) = cz_1'.$$

Hence we see from (42) that $g = l_1$, and from (10) that $l_{,1}$ vanishes, and then from (13) that $l_1 = 1 = g$; so that (42) becomes

$$l_{,} = r^2(1 + cz'). \tag{44}$$

Now assume $x = h\cos v$, $y = h\sin v$, and $z = kv$. Then we easily obtain the following equations:

$$\left.\begin{aligned}\frac{ds}{v} &= \sqrt{h^2+k^2}, & x' &= \frac{-y}{\sqrt{h^2+k^2}},\\ y' &= \frac{x}{\sqrt{h^2+k^2}}, & z' &= \frac{k}{\sqrt{h^2+k^2}}.\end{aligned}\right\} \tag{45}$$

$$x'' = \frac{-x}{h^2+k^2}, \qquad y'' = \frac{-y}{h^2+k^2}, \qquad z'' = 0. \tag{46}$$

From (46) and the second of equations (1) we find

$$r = \frac{h^2+k^2}{h}. \tag{47}$$

Hence (44) becomes

$$l_{,} = r^2\left\{1 + \frac{ck}{\sqrt{h^2+k^2}}\right\};$$

and since $a$, $b$, $f_{,}$ and $f_{,,}$ are zero, equation (4) becomes

$$r^2\left\{1 + \frac{ck}{\sqrt{h^2+k^2}}\right\}\frac{h^2}{(h^2+k^2)^{\frac{3}{2}}} = -f, \tag{48}$$

while either of equations (5) gives

$$r^2\left\{1 + \frac{ck}{\sqrt{h^2+k^2}}\right\}\frac{k}{(h^2+k^2)^{\frac{3}{2}}} = c. \tag{49}$$

Substituting for $r^2$ in (49) its value, that equation becomes

$$k\sqrt{h^2+k^2} = c(h^2+k^2); \tag{50}$$

so that we have

$$\sqrt{h^2+k^2} = \frac{ch^2}{k} - ck.$$

Whence

$$\frac{c}{k} = \frac{\sqrt{h^2+k^2}}{h^2-k^2}. \tag{51}$$

Next substituting the value of $r^2$ in (48), it becomes

$$-f=\sqrt{h^2+k^2}+ck=\frac{ch^2}{k}=h^2\frac{\sqrt{h^2+k^2}}{h^2-k^2}. \qquad (52)$$

From equations (50) and (52) we see that we cannot have $h$ and $k$ equal; but with this exception the assumptions $x = h \cos v$, $y = h \sin v$ and $z = kv$ will satisfy all the conditions of the question, and the helix will therefore be the solution required.

**351.** When problems of relative maxima or minima are to be considered, the same method must be adopted as in the case of two co-ordinates; that is, we multiply the integral which is to remain constant by a constant, say $a$; and it seems, therefore, unnecessary to introduce here any question of this class. Indeed, as the method of treating all the problems which belong to this section, whether of absolute or relative maxima and minima, is quite uniform, our knowledge of the calculus of variations would not be materially increased by their multiplication. Moreover, these questions generally lead us into work of considerable length, and rarely afford us any solution in finite terms, and are therefore somewhat wearisome. We shall therefore merely state two or three additional problems which the reader will find in the work by Prof. Jellett, or in the more recent French work, *Calcul des Variations*, by Moigno and Lindelöf.

(1) To draw between two fixed points or curves upon a given surface a curve which will maximize or minimize the expression

$$U=\int_{s_0}^{s_1}(v+Vx')\,ds,$$

$v$ and $V$ being functions of the co-ordinates $x$, $y$ and $z$ only.

(2) Two fixed points on a surface being given, and a curve connecting them, it is required to draw between these points a curve of given length such that the portion of the given

surface included between the given and the required curve may be a maximum.

(3) To find the form which a cord resting upon a given surface must assume in order that its centre of gravity may be as low as possible.

**352.** It will readily appear that while the adoption of $s$ as the independent variable often presents great advantages in the discussion of the terms of the first order, it is exceedingly unfavorable to a successful examination of those of the second order. For, in the first place, even when the limiting values of $x, y, z$, etc., are fixed, we would be obliged to add to $\delta U$ the terms

$$V_1 ds_1 - V_0 ds_0 + \frac{1}{2}\left\{\left(\frac{dV}{ds}\right)_1 ds_1^2 - \left(\frac{dV}{ds}\right)_0 ds_0^2\right\},$$

and then the relations between $ds$ and $\delta x$, $\delta y$ and $\delta z$ at either limit, which we have previously used, and which are true to the first order only, must be replaced by more accurate equations. In the use of these more accurate equations, certain terms of the second order will evidently arise at the limits; and as we may only equate those of the first order to zero, these terms cannot be neglected, but must be added to those already in the second order, thus rendering them more complicated.

In the second place, when we are obliged to use the method of Lagrange, we must render that method true to the second order, which we have not hitherto done. To accomplish this, whether $x$ or $s$ be the independent variable, we first take the variation of $U$ to the second order. Then, supposing the connecting equation to be $f(x, y, z) = 0 = f$, we shall have

$$\delta f = f_x \delta x + f_y \delta y + f_z \delta z$$
$$+ \frac{1}{2}(f_{xx}\delta x^2 + 2 f_{xy}\delta x \delta y + f_{yy}\delta y^2 + \text{etc.}) = 0.$$

Hence we may write $l\int \delta f\,dx = 0$, where the limits depend upon the independent variable.

Now after having given to $l$ such a value as will cause the terms of the first order to vanish after $l\int \delta f\,dx$ has been added to $\delta U$, we must remember that the variations in the terms of the second order are not independent, but are still connected by the equation $f = 0$.

If then these terms should be certainly invariably positive or negative, we have a minimum in the former and a maximum in the latter case. But as we shall generally be unable, if $f$ be a differential equation, to impose this restriction in any explicit manner upon the variations, we shall not usually be successful in determining the sign of these terms. Of course when $f$ is a differential equation, its variation is taken to the second order, as already explained.

**353.** In the discussion of problems involving three co-ordinates, we have, according to our usual method, ascribed no variation to the independent variable, whether that variable be $x$ or $s$. But it is quite common among writers to vary the independent variable also, just as has been already explained for problems of two co-ordinates.

Consider first, for a moment, the case in which $x$ is the independent variable. Here we follow without change the reasoning of Art. 264 until we arrive at equation (7), after which we still follow the article, only observing, in finding the values of $\left[\frac{dV}{dx}\right]$ and $\delta V$, that $V$ is now a function of $x, y, y', \ldots\ldots z, z', \ldots\ldots$ Then, having obtained the longer expression for $\delta U$, which will replace equation (10), it is evident that equations (15), Art. 265, will still hold true to the first order, and that, by the same reasoning for $x$ and $z$, we shall have the additional equations

$$\delta z' = \frac{d\omega^z}{dx} + z''\delta x, \qquad \delta z'' = \frac{d^2\omega^z}{dx^2} + z'''\delta x, \qquad \text{etc.}; \quad (1)$$

where $\omega^z = \delta z - z'\delta x$. Therefore, proceeding as in Art. 266, we shall obtain, instead of equation (17), an expression for $\delta U$ identical in form with that which would result from ascribing no variation to $x$, except that $\omega$ and $\omega^z$ will replace $\delta y$ and $\delta z$.

Moreover, since, to the first order, $\omega$ and $\omega^z$ equal $\delta y$ and $\delta z$ in the other method, we see that whatever relations may hold between these variations when the ordinary method is employed must hold also between $\omega$ and $\omega^z$ when $x$ is varied, so that, as in the case of two co-ordinates, the same general equations will be obtained by either method, and it will be found also that the same equations at the limits can be established by either method.

Next, when $s$ is the independent variable, we proceed as in Art. 296, merely observing, in finding the values of $v'$ and $\delta v$, that $v$ is now a function of $s$ and $x$, $y$ and $z$ with their differential coefficients with respect to $s$. Moreover, we shall have, in addition to equations (7) of that article, which will still hold true to the first order, the equations

$$\delta z' = (\omega^z)' + z''\delta s, \qquad \delta z'' = (\omega^z)'' + z'''\delta s, \qquad \text{etc.}, \quad (2)$$

where $\omega^z = \delta z - z'\delta s$.

Hence, as in the case of two co-ordinates only, we shall find that $\delta U$ will take the same form as if we had ascribed no variations to $s$, except that $\omega^x$, $\omega^y$ and $\omega^z$ will take the place of $\delta x$, $\delta y$ and $\delta z$ respectively.

# CHAPTER III.

## MAXIMA AND MINIMA OF MULTIPLE INTEGRALS.

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### SECTION I.

*CASE IN WHICH U IS A DOUBLE INTEGRAL; THE LIMITING VALUES OF X, Y, Z, ETC., BEING FIXED.*

### Problem LVII.

**354.** *Suppose we require the form of the surface of least area terminated in all directions by a certain fixed and closed linear boundary.*

If this boundary were a plane curve or any linear figure situated entirely in the same plane, the required surface would of course be itself plane. But we here wish that the bounding frame or edge may have any assigned form whatever, not inconsistent with the condition that it shall be closed.

Suppose, then, the required minimum surface to have been obtained, and call it the required surface, and suppose we take any other surface having a common edge with the first, and call this the derived surface. Then it will appear, as in Prob. I., that to prove the required surface to be that of least area we must, in the first place, assume that the derived surface with which its area is compared differs from it in form infinitesimally only. Then if the surface found have a less area than any such derived surface, it will be a

minimum, that term being used in the technical sense already explained, and it will then be in order, in discussing the least surface, to consider whether there may be any other minima.

We shall then at present discuss only the problem of finding the minimum surface.

**355.** Now let $x$, $y$ and $z$ be the co-ordinates of any point of the required surface, and suppose four indefinite planes—two parallel to the plane of $yz$, and two parallel to that of $xz$, the distance between the former two being $dx$, and between the latter two $dy$. Then, denoting by $ds$ an element of the surface intercepted at any point by any four planes drawn as above, it is known that we shall have

$$ds = \sqrt{1 + z'^2 + z_{\prime}^2}\, dy\, dx,$$

where accents above denote total differentiation with respect to $x$, and those below the same with respect to $y$. See De Morgan's Diff. and Integ. Calc., p. 444. Therefore, designating the surface whose area is to become a minimum by $U$, we have to minimize the expression

$$U = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \sqrt{1 + z'^2 + z_{\prime}^2}\, dy\, dx = \int_{x_0}^{x_1} \int_{y_0}^{y_1} V dy\, dx. \qquad (1)$$

**356.** It is essential that we should here recall from the theory of double integration a clear conception of the precise meaning of equation (1). Suppose, then, the entire surface to be divided into strips by planes parallel to that of $yz$, the distance between these planes being $dx$. Then, the area in question equals the sum of these strips, while that of any strip is itself equal to the sum of the elemental areas intercepted on it by successive planes parallel to that of $xz$, and separated by the distance $dy$.

To effect this latter summation, which we shall always suppose to have been first accomplished, we must imagine

the value of $V$ to have been obtained from the general equation of the surface, thus rendering $V$ some function of $x$ and $y$ only, since $z$ is some function of $x$ and $y$; and then, as $x$ and $dx$ will have the same value for every element of the same strip, while $y$ will vary, we must integrate the expression $Vdydx$ under the supposition that $x$ and $dx$ are constants. But since the summation must extend throughout any strip which we wish to consider, if we denote by $y_0$ and $y_1$ the values of $y$ at its extremities, the area of any strip will evidently be given by the expression $\int_{y_0}^{y_1} Vdydx$, $x$ and $dx$ being treated as constants. But because $V$ was made a function of $x$ and $y$ only, $\int Vdy$ will be a function of these quantities; and since for any particular strip $y_0$ and $y_1$ will certainly be functions of $x$ only, and perhaps constants, if we put $S$ for the area of any strip, we may write

$$S = f(x)dx = fdx. \tag{2}$$

Now to effect the summation of the strips, which is always the latter process, we suppose the edge or contour of the surface, when it has been projected upon the plane of $xy$, to form a curve capable of being expressed by the equation $y = F(x)$, which curve we shall call the *projected contour*. Then equation (2), which was before true for any strip, becomes so for every strip. Hence we need no longer regard $x$ as constant; and integrating from $x_0$ to $x_1$, where $x_0$ and $x_1$ denote the extreme abscissæ of the surface, or rather of the *projected contour*, we shall obtain the entire surface $U$.

**357.** Hitherto we have usually employed the suffixes 0, 1, etc., to denote what the quantity to which they are applied will become when the independent variable receives a particular value. Now because, in the discussion of curves, whether situated in space or not, we have but one independent vari-

able, $x$ or $s$ or some other, this method is satisfactory. But in problems relative to surfaces, where no curve is traced, $x$ and $y$ are evidently entirely independent, so that the substitution of a particular value of one variable does not necessitate the substitution of any particular value of the other, as it would if we were discussing a curve. It is, therefore, important that we should be able to specify just what substitutions of each variable are to be made in any function, which cannot be conveniently done by suffixes, particularly when we come to integrals of the third or higher order, involving three or more independent variables.

These substitutions are indicated in the following simple manner. Let $x, y, z$, etc., be any quantities whatever, and let $F$ be any function of these quantities. Then when we put for any of these quantities a particular value, as $x_1$ for $x$, we write $/^{x_1} F$, it being always understood that a suffixed quantity is substituted for the unsuffixed one of the same name. Also, if it be necessary to denote that $x_1$ has been substituted for $x$ and $y_1$ for $y$, we write the new function thus, $/^{x_1} /^{y_1} F$, where we shall always suppose $y_1$ to have been first substituted for $y$, after which $x_1$ is substituted for $x$ in the resulting function.

Again, suppose $f$ to be a function integrable with respect to $x$, its general integral being $F$. Then we may write

$$\int_{x_0}^{x_1} f\,dx = F_1 - F_0 = /^{x_1} F - /^{x_0} F.$$

Now as we shall often have expressions of a similar form arising from definite integration, we write the last equation thus, $\int_{x_0}^{x_1} f\,dx = /_{x_0}^{x_1} F$, where it will be always signified that we are to substitute successively the upper and lower suffixed for the corresponding unsuffixed quantity, and then subtract the second result from the first.

Extending this principle still further, $\Big/_{x_0}^{x_1}\Big/_{y_0}^{y_1} F$ will denote the following operations: first, that we must substitute successively $y_1$ and $y_0$ for $y$, and subtract the second result from the first; and second, that in the result we must substitute $x_1$ and $x_0$ for $x$, and subtract as before. Thus we shall have

$$\Big/_{x_0}^{x_1}\Big/_{y_0}^{y_1} F = \Big/_{x_0}^{x_1}\left\{\Big/^{y_1} F - \Big/^{y_0} F\right\}$$

$$= \Big/^{x_1}\Big/^{y_1} F - \Big/^{x_1}\Big/^{y_0} F - \Big/^{x_0}\Big/^{y_1} F + \Big/^{x_0}\Big/^{y_0} F. \qquad (3)$$

**358.** The idea of employing a sign to denote substitution is due to M. Sarrus, who calls it the sign of substitution, a name which we shall retain; and it seems probable that, as Prof. Todhunter has remarked, since Lagrange introduced his symbol $\delta$, nothing has been suggested which is of such service to the calculus of variations as this sign. But the sign and the method of employing it were subsequently modified by Cauchy, whose method we substantially follow, as exhibited in the *Calcul des Variations* by Moigno and Lindelöf.

**359.** As an illustration of the preceding discussion, let us suppose the given surface to be spherical, taking the origin at its centre, and considering only some portion of the upper hemisphere, whose edge or contour is to have any form we please. We may notice that $z'$ is the partial differential coefficient of $z$ with respect to $x$, and is obtained therefore from the equation of the surface by regarding $y$ as constant; and similarly $x$ must be constant in obtaining $z_{,}$. The equation of the sphere is $x^2 + y^2 + z^2 = r^2$. Whence

$$z' = -\frac{x}{z} = \frac{-x}{\sqrt{r^2 - x^2 - y^2}}, \qquad z_{,} = -\frac{y}{z},$$

$$\sqrt{1 + z'^2 + z_{,}^2} = \frac{r}{\sqrt{r^2 - x^2 - y^2}};$$

so that (1) becomes

$$U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{r\,dy\,dx}{\sqrt{r^2 - x^2 - y^2}}.$$

But regarding $x$ as constant, we have

$$\int \frac{r\,dy}{\sqrt{r^2 - x^2 - y^2}} = r\sin^{-1}\frac{y}{\sqrt{r^2 - x^2}} + c;$$

and the definite integral may be written

$$S = \int_{y_0}^{y_1} r\sin^{-1}\frac{y}{\sqrt{r^2 - x^2}}\,dx.$$

Thus we see that $S$ does not contain $z$, and is also independent of the general values of $y$, but may still be some function of $x$.

Now if we wish to denote the area of any particular strip for which $x = x_a$, we have only to write

$$\int^{x_a}\int_{y_0}^{y_1} r\sin^{-1}\frac{y}{\sqrt{r^2 - x^2}}\,dx.$$

To complete the integration, let us require all the surface for which neither $x, y$ nor $z$ shall become negative. Then we shall have

$$y_0 = 0,\ y_1 = \sqrt{r^2 - x^2},\ S = \frac{\pi r\,dx}{2},\quad U = \int_{x_0}^{x_1}\frac{\pi r\,dx}{2} = \Big/_{x_0}^{x_1}\frac{\pi r x}{2};$$

and since the entire eighth of the sphere is required, $x_0 = 0$ and $x_1 = r$, and $U = \dfrac{\pi r^2}{2}$.

**360.** Returning now to our original problem, we see that we can pass from any given surface to any other differing from it infinitesimally in form, and having a common edge, by

giving to $z$ suitable infinitesimal increments throughout the surface, the values of both $x$ and $y$ undergoing no change; and as $dz$ indicates the change which $z$ undergoes when we pass from one point to its consecutive on the same surface, we designate the new increments, as before, by $\delta z$. Moreover, we can also, without varying $x$ or $y$, obtain the derived surface by giving infinitesimal variations to $z'$ and $z_{,}$, which are the tangents of the angles made with the plane of $xy$ by those two edges of any elemental area which meet at the point $x, y, z$.

If now we denote by $\delta U$ the change in area which the entire surface will undergo when $z, z'$ and $z_{,}$ receive infinitesimal variations, the required surface must evidently be such as to render $\delta U$ negative. But as we cannot express $U$ in any more explicit form than that given in (1), and as we must compare the required surface with such as can be derived by infinitesimal changes in its form, we are compelled to seek the variation of the double integral in (1) in order to determine what conditions will render the variation negative.

**361.** In order to consider the subject more generally, let us assume the equation

$$U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} V\,dy\,dx, \tag{1}$$

where $V$ is any function of $x, y, z, z'$ and $z_{,}$, the limiting values of $x, y$ and $z$ being fixed; and let us, for convenience, write $z' = p$, $z_{,} = q$, $z'' = r$, $z_{,}' = s$ and $z_{,,} = t$. Then if we change $z$ into $z + \delta z$, $p$ into $p + \delta p$ and $q$ into $q + \delta q$, $x$, $dx$, $y$ and $dy$ remaining unaltered, and denote by $\delta U$ and $\delta V$ the changes which $U$ and $V$ will undergo, we shall have

$$U + \delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} (V + \delta V)\,dy\,dx$$

$$= \int_{x_0}^{x_1}\int_{y_0}^{y_1} V\,dy\,dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1} \delta V\,dy\,dx.$$

Whence, from (1),

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \delta V \, dy \, dx. \tag{2}$$

We have now merely to determine $\delta V$ by Taylor's Theorem, which, since $x$ and $y$ undergo no change, will give

$$\delta V = V_z \delta z + V_p \delta p + V_q \delta q + \frac{1}{2}\Big\{ V_{zz}\delta z^2 + 2V_{zp}\delta z\,\delta p$$

$$+ V_{pp}\delta p^2 + 2V_{zq}\delta z\,\delta q + 2V_{pq}\delta p\,\delta q + V_{qq}\delta q^2 \Big\} + \text{etc.}; \tag{3}$$

where the etc. denotes terms of the third and higher orders, and the differentials of $V$ are all partial.

**362.** Now denoting by $S$ the terms of the second order in $\delta V$, with the exception of the $\frac{1}{2}$, we shall have

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \{ V_z \delta z + V_p \delta p + V_q \delta q \} dy \, dx$$

$$+ \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1} S \, dy \, dx + \text{etc.} \tag{4}$$

If now we require that $U$ shall become either a maximum or a minimum, it will, since $\delta z$, $\delta p$ and $\delta q$ are entirely in our power and may have either sign, appear, by precisely the same reasoning as in the case of single integrals, that the first integral in (4) must vanish, while the second must become invariably positive for a minimum and negative for a maximum.

Now we must observe that $x$ and $y$ are completely independent, and that $z'$ and $z''$ or $p$ and $r$ are the differential coefficients of $z$ with respect to $x$, $y$ being regarded as constant;

that is, in finding them, we regard $z$ as a function of $x$ only, and constants, among which we reckon $y$. Or we may regard $z$ as the ordinate of the curve made by the intersection of the required surface with a plane parallel to that of $xz$ at the distance $y$. Similarly, $z_{\prime}$ and $z_{\prime\prime}$ or $q$ and $t$ are the differentials of $z$ with respect to $y$, $x$ being constant; that is, $z$ may now be regarded as the ordinate of the section cut by a plane at right angles to the first, and at the distance $x$ from the plane of $yz$. Therefore, as $x$ and $y$ receive no variation, we we must have, as heretofore,

$$\delta z' \text{ or } \delta p = \frac{d\delta z}{dx}, \qquad \delta z'' \text{ or } \delta r = \frac{d^2\delta z}{dx^2},$$

$$dz_{\prime} \text{ or } \delta q = \frac{d\delta z}{dy}, \qquad \delta z_{\prime\prime} \text{ or } \delta t = \frac{d^2\delta z}{dy^2};$$

and these equations, which are exact, may be used in any manner we find convenient in transforming $\delta U$.

**363.** Considering for the present the terms of the first order only, we have

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \{V_z\delta z + V_p\delta p + V_q\delta q\}\,dy\,dx = 0. \qquad (5)$$

But without entering upon any general discussion of the conditions which must hold in order that (5) may be satisfied, let us return to our original problem. Here

$$V = \sqrt{1+p^2+q^2}, \qquad V_z = 0,$$

$$V_p = \frac{p}{\sqrt{1+p^2+q^2}}, \qquad V_q = \frac{q}{\sqrt{1+p^2+q^2}};$$

so that (5) gives

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\left\{\frac{p}{\sqrt{1+p^2+q^2}}\delta p + \frac{q}{\sqrt{1+p^2+q^2}}\delta q\right\}dy\,dx = 0. \quad (6)$$

Now we cannot assert that every element of this integral must vanish, because we have also required that the edges of the surface shall be fixed—that is, that $\delta z$, for all points of the edge or contour, shall vanish—and this condition has not yet been imposed upon $\delta U$. Indeed, there is an analogy between the present problem and Prob. I. For in Prob. I. we were to connect two fixed points by a line of minimum length, requiring us to minimize a single integral; while in the present problem we are to connect an infinite number of fixed points, forming the given contour, by a surface of minimum area, requiring thereby the minimizing of a double integral.

**364.** The condition just mentioned may be imposed somewhat as in Prob. I. For we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{q}{\sqrt{1+p^2+q^2}}\delta q\,dy\,dx$$

$$= \int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\frac{q}{\sqrt{1+p^2+q^2}}\delta z\,dx$$

$$- \int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2}}\delta z\,dy\,dx. \quad (7)$$

If now we remember that for any abscissa $x$, $y_0$ and $y_1$ are the two ordinates of the projected contour corresponding to this abscissa, we shall see that the $z$'s corresponding to $y_0$ and $y_1$ relate to the edge or contour only of the required surface, and that therefore every $\delta z$ in the single integral in (7) must vanish, causing the integral itself to vanish.

Now since we may adopt either order of integration in a

double definite integral without affecting its value, we may write

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{p}{\sqrt{1+p^2+q^2}}\,\delta p\,dy\,dx=\int_{y_0}^{y_1}\int_{x_0}^{x_1}\frac{p}{\sqrt{1+p^2+q^2}}\,\delta p\,dx\,dy$$

$$=\int_{y_0}^{y_1}\Big/_{x_0}^{x_1}\frac{p}{\sqrt{1+p^2+q^2}}\,\delta z\,dy$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{d}{dx}\frac{p}{\sqrt{1+p^2+q^2}}\,\delta z\,dy\,dx. \quad (8)$$

Here we regard $y$ as the independent variable in the equation of the *projected contour*, so that $x_0$ and $x_1$ are always ordinates of this contour, $y$ being the abscissa. Hence, as before, every $\delta z$ in the single integral of the last equation refers to some portion of the contour only, and must vanish.

Hence, finally, we must have

$$\delta U=\int_{x_0}^{x_1}\int_{y_0}^{y_1}-\left\{\frac{d}{dx}\frac{p}{\sqrt{1+p^2+q^2}}+\frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2}}\right\}\delta z\,dy\,dx$$

$$=\int_{x_0}^{x_1}\int_{y_0}^{y_1}M\,\delta z\,dy\,dx=0. \quad (9)$$

**365.** It is here necessary to notice two points.

First. It will be seen that the form in which the terms under the sign of single integration—which terms are the terms at the limits in this problem—have been left is incongruous, inasmuch as we do not retain the same independent variable throughout. But our only object at present is to show that when the contour is fixed the terms at the limits will vanish. Indeed, the arrangement of these terms in the case of multiple integrals, so as to enable us to discuss with anything like generality the conditions which must hold at the

limits, has proved to mathematicians one of the most difficult points connected with the calculus of variations. For although this subject had more or less occupied the attention of Gauss, Poisson, Ostrogradsky, Jacobi and Delaunay, the last of whom has been followed by Prof. Jellett, it remained for M. Sarrus to present a method of treatment which has the merit of being systematic and general, and is perhaps as nearly perfect as the nature of the subject will permit.

Second. The two differentials in (9) denote the entire change produced in the first fraction when we change $x$ into $x + dx$, and in the second when we change $y$ into $y + dy$, $p$ and $q$ being variable both for changes in $x$ and $y$, so that, with respect to $x$ or $y$ only, these differentials may be said to be total. Such differentials are, however, called partial, since they denote the change incident upon an alteration in one independent variable only, while there are two which might be varied.

**366.** Now the two single integrals in (7) and (8), taken together, certainly involve every $\delta z$ for the contour, which would not perhaps be true of the first integral alone if a portion of the *projected contour* should be a right line perpendicular to the axis of $x$, nor of the second if a portion of the *projected contour* should be a right line perpendicular to the axis of $y$. Hence in (9) the condition that $\delta z$ shall be zero throughout the entire contour has been imposed.

Now as the sign of $\delta z$ for every point of the required surface is wholly within our power, and its value is subject to no other restrictions than that it shall be infinitesimal, and shall render $\delta p$ and $\delta q$ also infinitesimal, it will appear, as hitherto, that we can only satisfy (9) by equating $M$ to zero, so that we shall have

$$\frac{d}{dx}\frac{p}{\sqrt{1+p^2+q^2}} + \frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2}} = -M = 0. \qquad (10)$$

Performing the differentiation indicated in the last equation, observing that $p_{,} = q' = s$, we have, after reducing to a common denominator,

$$\frac{(1+q^2)r - 2pqs + (1+p^2)t}{\sqrt{(1+p^2+q^2)^3}} = 0. \qquad (11)$$

This expression, which is a partial differential equation of the second order, is known to indicate that the required surface must be of such a nature that at every point the principal radii of curvature may be equal and taken with a contrary sign, so that their sum may be always zero. Moreover, equation (10), which is the fundamental equation, would evidently be satisfied by a plane, since $p$ and $q$ would then become constant. This could not, however, as we have already shown, be the general solution, because, if the given contour were not a plane figure it would not be possible to make a plane surface fulfil all the conditions at the limits; that is, to pass through every point of the given contour. But we shall resume the consideration of (11) presently.

**367.** Assuming the required surface in any particular case to have been determined, let us now examine the sign of the terms of the second order. Since $c$ does not enter $V$ explicitly, we have, from (2) and (4),

$$S = V_{pp}\delta p^2 + 2V_{pq}\delta p\,\delta q + V_{qq}\delta q^2$$

$$= \frac{(1+q^2)\,\delta p^2 - 2pq\,\delta p\,\delta q + (1+p^2)\,\delta q^2}{(1+p^2+q^2)^{\frac{3}{2}}}.$$

Whence, since the terms of the first order vanish, we may write

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{\delta p^2 + \delta q^2 + (q\delta p - p\delta q)^2}{(1+p^2+q^2)^{\frac{3}{2}}}\,dy\,dx, \qquad (12)$$

which shows that $\delta U$ is invariably positive, since every element of the double integral is essentially so.

We see, therefore, without solving (10), that so long as the contour of the required surface is to be fixed, any surface which satisfies (10) or (11), and can also pass through every point of the given contour, will possess a minimum area. We should not, however, say that the surface thus found is necessarily that of the least area. For although this may be true in the present problem, the method of variations does not of itself warrant the assertion. This will at once appear if we remember that the calculus of variations permits us to compare the required primitive surface with such derived surfaces only as differ from it infinitesimally in form; and we cannot, therefore, be certain that there might not be some other minimum surface whose area, being less, might itself be the least possible.

Moreover, since $\delta p$ and $\delta q$ must be infinitesimal, we are not permitted to consider any step-shaped surface; and one of these might, perhaps, be that of least area. In fact, it will appear that, theoretically at least, the distinction between maxima and minima, and greatest and least, values must hold equally whether the integral be single or double.

**368.** Let us now return to the terms of the first order. It is easy to see that had the equation been

$$U=\int_{x_0}^{x_1}\int_{y_0}^{y_1} V\,dy\,dx,$$

where $V$ contained $x$, $y$, $z$, $p$ and $q$, the same reasoning by which we obtained (10) would have given us the equation

$$M = V_z - (V_p)' - (V_q)_{,} = 0; \tag{13}$$

and from the case of single integrals we would naturally infer, what we shall presently show, that this equation will be true

independently of any conditions which may be required to hold at the limits.

Indeed, this fundamental equation appears to present itself naturally, and to have been obtained almost as soon as the subject was discussed; while no subsequent researches have given us any other equation. Now (13) will be what is known as a partial differential equation of the second order, the variables $x$ and $y$ being entirely independent, and $z$ being supposed to be some function of these variables. But the theory of such equations is still imperfect, it being uncertain even that every partial differential equation of the second order has any solution at all; so that we are very rarely able to obtain the complete integral of the equation $M = 0$, or indeed to obtain any solution whatever in finite terms.

We know, however, that when a partial differential equation of any order can be integrated completely, the integration will introduce certain arbitrary functions instead of the ordinary arbitrary constants, and that, however the solution be obtained, the number of these arbitrary functions will not exceed that of the order of the partial differential equation.

**369.** According to Moigno, the integral of equation (10) was first obtained by Monge, but in a form which rendered it of little use. Strictly speaking, however, this integral was not obtained by Monge in any form, but merely indicated. (See Monge, section on "The surface whose principal radii of curvature are equal, but with contrary sign"—Section XX. in Dr. Liouville's edition.)

The same integral was, according to Moigno, considered also by Legendre, and later by Messrs. Seret and Catalan, without obtaining any better results. Finally, however, M. Ossian Bonnet in an article on "The Employment of a New System of Variables," published in the fifth volume of the *Journal de Liouville*, 1860, has shown that the equation of the required surface is included under a still more general obtain-

able integral of comparative simplicity. We present in a note Bonnet's method, following the guidance of Moigno, and supplying formulæ and references, all of which he has omitted.

**370.** It appears that we can have an infinite number of surfaces, all satisfying equation (10), but it is evident that, when the contour is given, we are restricted to that surface or those surfaces which pass through every point of the fixed contour, and have at the same time their principal radii of curvature equal and of contrary sign. Or following the analogy of single integrals, if we suppose the general integral of equation (10) to have been obtained, we must so determine the arbitrary functions which arise in the integration as to cause the surface to pass through every point of the given contour. But as we are unable to present the integral of (10) in an available form, we cannot give anything more than this general outline of the treatment of the functions for this problem.

## Problem LVIII.

**371.** *Let $v$ be the portion of the axis of $z$ comprised between the origin and any tangent plane to a surface. Then it is required to determine the form of the surface which will minimize the expression*

$$U=\int_{x_0}^{x_1}\int_{y_0}^{y_1} v^m\,dy\,dx=\int_{x_0}^{x_1}\int_{y_0}^{y_1} V dy\,dx,$$

*the edges of the surface being, as before, confined to a fixed curve.*

It is well known that this intersept is

$$v = z - px - qy,$$

so that we have

$$U=\int_{x_0}^{x_1}\int_{y_0}^{y_1}(z-px-qy)^m\,dy\,dx=\int_{x_0}^{x_1}\int_{y_0}^{y_1} V dy\,dx. \qquad (1)$$

Here, then, we have

$$\left.\begin{aligned} V_z = mv^{m-1}, \quad V_p = -mv^{m-1}x, \quad V_q = -mv^{m-1}y, \\ (V_p)' = -mv^{m-1} - xm(m-1)v^{m-2}v', \\ (V_q)_{,} = -mv^{m-1} - ym(m-1)v^{m-2}v_{,}\,; \end{aligned}\right\} \quad (2)$$

so that the fundamental equation

$$V_z - (V_p)' - (V_q)_{,} = 0$$

becomes, after dividing by $m(m-1)$,

$$v^{m-2}\left\{\frac{3v}{m-1} + xv' + yv_{,}\right\} = 0, \qquad (3)$$

the terms at the limits vanishing as before, because they involve the values of $\delta z$ for the contour only, which are all zero.

Now one solution of (3) is evidently $v = 0$; which gives $U = 0$, and signifies that the surface is a conic surface, having its apex at the origin, as all its tangent planes must pass through that point. But neglecting this supposition, which is only a singular solution, and will evidently not answer for all supposable contours, we shall have

$$\frac{3v}{m-1} + xv' + yv_{,} = 0. \qquad (4)$$

**372.** Equation (4), although in reality an equation of the second order, is a partial differential equation of the first order in $v$; and being written under the form $xv' + yv_{,} = \frac{3v}{1-m}$, is easily integrated by the ordinary method for such equations (see De Morgan's Diff. and Integ. Calc., p. 203, where we put $X = x$, $Y = y$, $z = u\,\frac{3v}{1-m} = U$), and gives

$$v = x^{\frac{3}{1-m}} f\left(\frac{y}{x}\right), \qquad (5)$$

where $f$ denotes any function whatever of $\frac{y}{x}$. Substituting the value of $v$, (5) may be written

$$xp + yq = z - x^{\frac{3}{1-m}} f\left(\frac{y}{x}\right). \tag{6}$$

Now by putting the second member of this equation for $U$ in De Morgan, (6) can be integrated by the same method as before, and writing

$$\frac{m-1}{m+2} f\left(\frac{y}{x}\right) = F\left(\frac{y}{x}\right),$$

we obtain

$$z = x^{\frac{3}{1-m}} F\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right), \tag{7}$$

where $F$ and $f'$ are any functions whatever.

**373.** Thus we have been able in the present case to obtain the general integral of the equation $M = 0$. This integral represents an infinite number of surfaces according to the forms which we assign to the functions $F$ and $f'$. If, as hitherto, we suppose the contour of the required surface to be some linear boundary fixed in space, we must so determine the forms of these arbitrary functions as to cause the surface to pass through every point of this boundary. But we shall consider the determination of these functions more fully hereafter.

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## SECTION II.

### *FORMULÆ NECESSARY FOR THE TRANSFORMATION OF THE VARIATION OF A MULTIPLE INTEGRAL.*

**374.** We propose in the present section to present some formulæ which belong, strictly speaking, to the differential and integral calculus only, but which, having been developed

by M. Sarrus for the purpose of so transforming the variation of a multiple integral as to enable us to discuss more satisfactorily the conditions which must hold at the limits, will not be found in treatises on the ordinary calculus. We begin with a formula which is generally known.

**375.** Assume the equation $U=\int_{x_0}^{x_1} u\,dx$, where $u$ is some function of $x$, $t$, etc., $t$ being either a constant, or a variable which is to be regarded as a constant in obtaining the integral. If we change $t$ into $t+\delta t$, we shall, in the most general case which can arise, have

$$\delta U = u_1\,dx_1 - u_0\,dx_0 + \int_{x_0}^{x_1} \frac{du}{dt}\,\delta t\,dx.$$

For since $t$ is a constant, it is independent of the general values of $x$, so that we may vary it without varying $x$. But the values of $x_1$ and $x_0$ are constants, and may, or may not, be independent of $t$. In the former case the first two terms of the last equation will vanish, but in the latter they must evidently be retained. But it is evidently immaterial in the last equation whether we employ the symbol $\delta$ or $d$; so that if we regard $dt$ as an infinitesimal constant, we may write

$$\frac{d}{dt}\int_{x_0}^{x_1} u\,dx = \int_{x_0}^{x_1} \frac{du}{dt}\,dx + \frac{u_1 dx_1}{dt} - \frac{u_0 dx_0}{dt}.$$

Or using the sign of substitution, already explained, we have

$$\frac{d}{dt}\int_{x_0}^{x_1} u\,dx = \int_{x_0}^{x_1} \frac{du}{dt}\,dx + \Big/_{x_0}^{x_1} u\,\frac{dx}{dt}; \tag{1}$$

where it will be seen that we make the sign of substitution mean also that $dx_1$ and $dx_0$ are put for $dx$; and this will be always the case except it be otherwise indicated. As for $dt$, because it is constant, $dt_1$ differs in no respect from $dt$, and it

is immaterial whether we consider it as controlled by the sign of substitution or not.

**376.** Again, assume $u$, as before, to be any function of $x$, $t$, etc., where $t$ may be any quantity independent of, or dependent upon, $x$. Then, in the most general case, we must have

$$\left[\frac{du}{dt}\right] = \frac{du}{dt} + \frac{du}{dx}\frac{dx}{dt}.$$

Hence, if we wish to consider $u$ when $x$ has some particular value, as $x_1$, $u$ will become $/^{x_1} u$, and we may write

$$\frac{d}{dt}/^{x_1} u = /^{x_1} \left\{ \frac{du}{dt} + \frac{du}{dx}\frac{dx}{dt} \right\}. \tag{2}$$

These formulæ, which are simple enough, will be found to be of the highest importance as we proceed.

**377.** Again, we evidently have

$$\int_{x_0}^{x_1} \frac{du}{dx}\, dx = /_{x_0}^{x_1} u. \tag{A}$$

Now change $u$ into $\int_{y_0}^{y_1} u\, dy$. Then we have from (A), writing $dx$ first to prevent confusion,

$$\int_{x_0}^{x_1} dx \frac{d}{dx}\int_{y_0}^{y_1} u\, dy = /_{x_0}^{x_1} \int_{y_0}^{y_1} u\, dy.$$

But from (1) we have

$$\frac{d}{dx}\int_{y_0}^{y_1} u\, dy = \int_{y_0}^{y_1} \frac{du}{dx}\, dy + /_{y_0}^{y_1} u\frac{dy}{dx};$$

because when we integrate with respect to $y$, $x$ is regarded as constant, and we may therefore, since $y$ is the independent variable, put it in the place of $x$, and $x$, which is now constant, in the place of $t$. We may evidently employ (1) in like manner for any other variables, replacing $x$ by that variable, which is then to be regarded as independent, and $t$ by that which is then to be regarded as constant in the integration. Hence, by substitution, we have

$$\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} \frac{du}{dx}\,dy + \int_{x_0}^{x_1} dx \Big/_{y_0}^{y_1} u\,\frac{dy}{dx} = \Big/_{x_0}^{x_1} \int_{y_0}^{y_1} u\,dy. \tag{B}$$

Now in this equation change $u$ into $ut$. Then, observing that the integral signs in any term, having been separated merely to better indicate the distance to which each extends, may be again brought together, as may also the differentials, we shall have, since $\frac{d}{dx}\,ut = \frac{udt}{dx} + \frac{du}{dx}\,t$,

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} u\,\frac{dt}{dx}\,dy\,dx = -\int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{du}{dx}\,t\,dy\,dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} ut\,dy$$

$$-\int_{x_0}^{x_1}\Big/_{y_0}^{y_1} ut\,\frac{dy}{dx}\,dx. \tag{3}$$

This formula would evidently enable us to transform such a term as $\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_p\,\delta p\,dy\,dx$.

**378.** Again, if in (A) we change $u$ into $\Big/^{y_1} u$, we shall have

$$\int_{x_0}^{x_1} \frac{d}{dx} \Big/^{y_1} u\,dx = \Big/_{x_0}^{x_1} \Big/^{y_1} u. \tag{C}$$

But from (2), putting $x$ for $t$ and $y$ for $x$, we have

$$\frac{d}{dx}\Big/^{y_1} u = \Big/^{y_1} \left\{ \frac{du}{dx} + \frac{du}{dy}\,\frac{dy}{dx} \right\}.$$

Therefore we have, by substituting in (C),

$$\int_{x_0}^{x_1} dx \Big/^{y_1} \left\{ \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} \right\} = \Big/_{x_0}^{x_1} \Big/^{y_1} u.$$

Now if in this equation we, as before, change $u$ into $ut$, we shall obtain

$$\int_{x_0}^{x_1} \Big/^{y_1} u \frac{dt}{dx}\, dx = -\int_{x_0}^{x_1} \Big/^{y_1} \frac{du}{dx}\, t\, dx + \Big/_{x_0}^{x_1} \Big/^{y_1} ut$$

$$-\int_{x_0}^{x_1} \Big/^{y_1} u \frac{dt}{dy}\frac{dy}{dx}\, dx - \int_{x_0}^{x_1} \Big/^{y_1} \frac{du}{dy}\, t \frac{dy}{dx}\, dx. \qquad (4)$$

**379.** We have also the equation

$$\int_{y_0}^{y_1} \frac{du}{dy}\, dy = \Big/_{y_0}^{y_1} u. \qquad \text{(D)}$$

Hence

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \frac{du}{dy}\, dy\, dx = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} \frac{du}{dy}\, dy = \int_{x_0}^{x_1} dx \Big/_{y_0}^{y_1} u\,;$$

and changing $u$ into $ut$, we have

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} u \frac{dt}{dy}\, dy\, dx = -\int_{x_0}^{x_1} \int_{y_0}^{y_1} \frac{du}{dy}\, t\, dy\, dx + \int_{x_0}^{x_1} \Big/_{y_0}^{y_1} ut\, dx. \qquad (5)$$

**380.** Again, from (D), we have

$$\Big/^{x_1} \int_{y_0}^{y_1} \frac{du}{dy}\, dy = \Big/^{x_1} \Big/_{y_0}^{y_1} u. \qquad \text{(E)}$$

Whence, changing, as usual, $u$ into $ut$, we obtain

$$\Big/^{x_1} \int_{y_0}^{y_1} u \frac{dt}{dy}\, dy = -\Big/^{x_1} \int_{y_0}^{y_1} \frac{du}{dy}\, t\, dy + \Big/^{x_1} \Big/_{y_0}^{y_1} ut. \qquad (6)$$

We need hardly say that formulæ (2), (4) and (6) will often be replaced by others, in which the suffixes of those quantities which are written above the sign of substitution only will be changed from 1 to 0, everything else remaining unaltered; and observing this fact, equations (3), (4), (5) and (6) are sufficient for the transformation of the variation of a double integral.

**381.** It will, however, be convenient to deduce also in this place the formulæ necessary for the transformation of the variation of a triple integral. But the reader is advised to omit these formulæ for the present, returning to them when they are required.

**382.** If in equation (B) we change $u$ into $\int_{z_0}^{z_1} u\,dz$, and separate the integrals as before, we shall have

$$\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy \frac{d}{dx} \int_{z_0}^{z_1} u dz =$$

$$-\int_{x_0}^{x_1} dx \Big/_{y_0}^{y_1} \frac{dy}{dx} \int_{z_0}^{z_1} u\,dz + \Big/_{x_0}^{x_1} \int_{y_0}^{y_1} dy \int_{z_0}^{z_1} u\,dz.$$

But from (1) we have

$$\frac{d}{dx}\int_{z_0}^{z_1} u\,dz = \int_{z_0}^{z_1} \frac{du}{dx}\,dz + \Big/_{z_0}^{z_1} u\frac{dz}{dx}.$$

Whence, by substituting and reuniting the integral signs, we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \frac{du}{dx}\,dz\,dy\,dx = -\int_{x_0}^{x_1}\int_{y_0}^{y_1} \Big/_{z_0}^{z_1} u\frac{dz}{dx}dy\,dx$$

$$-\int_{x_0}^{x_1} \Big/_{y_0}^{y_1} \frac{dy}{dx} \int_{z_0}^{z_1} u\,dz\,dx + \Big/_{x_0}^{x_1} \int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dy.$$

Now changing $u$ into $ut$, we shall obtain

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\frac{dt}{dx}\,dz\,dy\,dx =$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\frac{du}{dx}\,t\,dz\,dy\,dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} ut\,dz\,dy$$

$$-\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\frac{dy}{dx}\int_{z_0}^{z_1} ut\,dz\,dx - \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} ut\frac{dz}{dx}\,dy\,dx. \qquad (7)$$

**383.** Now it is evident that equation (B) will hold for any two independent variables involved in $u$. Let us therefore put $y$ for $x$, and $z$ for $y$. Then we shall have

$$\int_{y_0}^{y_1}\int_{z_0}^{z_1}\frac{du}{dy}\,dz\,dy = -\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} u\frac{dz}{dy}\,dy + \Big/_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz.$$

Therefore, since the integrals are all definite, we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\frac{du}{dy}\,dz\,dy\,dx =$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} u\frac{dz}{dy}\,dy\,dx + \int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dx.$$

Now change $u$ into $ut$, and we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\frac{dt}{dy}\,dz\,dy\,dx = -\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\frac{du}{dy}\,t\,dz\,dy\,dx$$

$$+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1} ut\,dz\,dx - \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} ut\frac{dz}{dy}\,dy\,dx. \qquad (8)$$

**384.** Again we have

$$\int_{z_0}^{z_1}\frac{du}{dz}\,dz = \Big/_{z_0}^{z_1} u.$$

Therefore, because the integrals are definite, we may write

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\frac{du}{dz}\,dz\,dy\,dx=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1}u\,dy\,dx.$$

If now we change $u$ into $ut$, we shall obtain

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}u\frac{dt}{dz}\,dz\,dy\,dx=$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\frac{du}{dz}\,t\,dz\,dy\,dx+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1}ut\,dy\,dx. \qquad (9)$$

The preceding formulæ will be found sufficient for the purposes of this work, although the notion of variations adopted by M. Sarrus, and the generality of the integrals which he proposes to consider, have caused him to develop to a much greater extent than we have done this department of the calculus, which might be termed the *calculus of substitution*. Although we shall subjoin without demonstration a few more formulæ when we come to explain Sarrus's notion of variations, the reader who wishes to find in a neat and compact form the various formulæ which may present themselves in this calculus of substitution is referred to the *Calcul des Variations*, by Moigno and Lindelöf, Leçons I. and II.

## SECTION III.

### *MAXIMA AND MINIMA OF DOUBLE INTEGRALS WITH VARIABLE LIMITS*

### Problem LIX.

**385.** *In either of the preceding problems, instead of supposing the contour to be a fixed boundary in space, let it be required merely that its projection upon the plane of xy—that is, the projected contour—shall be some fixed and closed boundary. In other words, let it be required that the contour shall always touch certain cylindrical walls of a given form.*

The terms cylindrical and conic must in this chapter be understood in their most general sense; the first denoting a surface generated by the movement of a right line which remains always parallel to a given line, and the second that generated by the movement of a right line which always passes through a given point. Here the walls are generated by the movement along the *projected contour* of a right line which remains always parallel to the axis of $z$.

It will readily appear that in this case the limiting values of $x$ and $y$ are still fixed, because they belong only to the *projected contour*, which is fixed, but that $z$ along the contour—that is, along the limiting walls—is susceptible of variation, so that $\delta z$ at the limits is no longer necessarily zero.

**386.** We are then led by the preceding problem to examine what will be the form of $\delta U$, where $U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} V\,dy\,dx$, $V$ being any function of $x$, $y$, $z$, $p$ and $q$, the limiting values of $r$ only being fixed. We have already seen that

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} (V_z\delta z + V_p\delta p + V_q\delta q)\,dy\,dx; \qquad (1)$$

and as that value of $\delta U$ was obtained under no other restriction than that the limiting values of $x$ and $y$ should be regarded as incapable of variation, it must hold in the case which we are considering.

If now, in equation (3), Art. 377, we write $u = V_p$, $t = \delta z$, $\dfrac{dt}{dx} = \dfrac{d\delta z}{dx} = \delta p$, we shall have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_p \delta p \, dy \, dx = -\int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{dV_p}{dx}\delta z \, dy \, dx$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_p \delta z \, dy - \int_{x_0}^{x_1}\int_{y_0}^{y_1} V_p \delta z \frac{dy}{dx} dx. \qquad (2)$$

Also if, in equation (5), Art. 379, we write $u = V_q$, $t = \delta z$, $t_{,} = \delta q$, we shall obtain

$$\int_{x_0}^{x_1}\int_{y_0}^{y_0} V_q \delta q \, dy \, dx =$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{dV_q}{dy}\delta z \, dy \, dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1} V_q \delta z \, dx. \qquad (3)$$

Now since the first term in the second member of (1) is not susceptible of any transformation, combining these results, observing that the substitution of a quantity in the sum, difference, product or quotient of two or more functions is the same as if we substituted in each function separately and *vice versa*, we may write

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \left\{ V_q - V_p \frac{dy}{dx} \right\} \delta z \, dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1} V_p \delta z \, dy$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1} \left\{ V_z - \frac{dV_p}{dx} - \frac{dV_q}{dy} \right\} \delta z \, dy \, dx, \qquad (4)$$

which must, of course, vanish if $U$ is to be a maximum or a minimum.

**387.** Now the *projected contour* need not be a continuous curve, but may be any combination of right lines, curved lines, or both, and we therefore speak of it as a boundary. Then $y_1$ and $y_0$ are the two ordinates of this boundary corresponding to any given value of $x$, and the substitution of $y_1$ or $y_0$ in any quantity causes that quantity to relate to the upper or the lower portion of this boundary only. To understand the effect of substituting $x_1$ or $x_0$ in any quantity, we observe that, whatever be the form of the *projected contour*, we must either have $y_0$ and $y_1$ zero, both when $x$ becomes $x_0$ and $x_1$, or it must consist in part of a right line perpendicular to the axis of $x$ at the point $x = x_0$ or $x = x_1$, or both. In other words, the *projected contour* will terminate in right lines whose equations are $x = x_0$ and $x = x_1$. In the latter case, then, the substitution of $x_0$ or $x_1$ in a quantity will cause it to relate to these right lines only, and in the former case, in which these lines may be regarded as becoming zero, the quantity will relate to the points $x_0$ and $x_1$ only, and will in general vanish.

**388.** Now writing $k = V_q - V_p\dfrac{dy}{dx}$, the first term of $\delta U$ in (4), when resolved, becomes

$$\int_{x_0}^{x_1} \Big/^{y_1} k\,\delta z\,dx - \int_{x_0}^{x_1} \Big/^{y_0} k\,\delta z\,dx, \tag{5}$$

in which the first integral is taken only along the entire intersection of the required surface with that portion of the cylindrical walls for which $y = y_1$, and the second along its intersection with those for which $y = y_0$.

The second term in $\delta U$, when resolved, becomes

$$\Big/^{x_1} \int_{y_0}^{y_1} V_p \delta z\,dy - \Big/^{x_0} \int_{y_0}^{y_1} V_p \delta z\,dy, \tag{6}$$

in which, although the integration is with respect to $y$, the first integral extends only along the right line $x = x_1$, while

the second extends only along the line $x = x_0$, and either integral will vanish of itself should the *projected contour* not terminate in the lines $x = x_0$ and $x = x_1$.

It appears, therefore, that the first two terms of $\delta U$ in (4) involve only values of $\delta z$ for the edge or contour of the required surface; and also that all these values are included. Now when, as in the present case, we require that the surface to be varied shall be comprised within certain cylindrical walls, the walls become the limit of the required surface, just as do two fixed lines $x = x_0$ and $x = x_1$, or two fixed planes with the same equations in an analogous problem for curves; so that the terms just considered in $\delta U$, although still affected by the integral sign, as indeed they ought to be in order that they may sum up the variations of $z$ for the entire contour, must be in the variation of a surface what the terms at the limits are in the variation of a curve.

**389.** Let us now examine what conditions must hold when $U$ is to be a maximum or a minimum.

Since $\delta U$ must now vanish, if we denote by $L$ the aggregate of the limiting terms, (4) may be written

$$\delta U = L + \int_{x_0}^{x_1}\int_{y_0}^{y_1} M\,\delta z\,dy\,dx = 0, \qquad (7)$$

where $M = V_z - (V_p)' - (V_q)_{,}$.

Now since the double integral extends throughout the entire surface, it will appear, as in the case of single integrals, that we cannot, without in some manner restricting the value of $\delta z$, make this general integral depend in any manner upon terms which refer solely to the limits, even when those terms are themselves under an integral sign. Therefore the terms in (7), being completely independent, must be equated severally to zero, so that we shall obtain, as before,

$$M = V_z - (V_p)' - (V_q)_{,} = 0. \qquad (8)$$

We shall also have, writing the value of $L$ from (5) and (6),

$$L = \int_{x_0}^{x_1} \Big/^{y_1} k\, \delta z\, dx - \int_{x_0}^{x_1} \Big/^{y_0} k\, \delta z\, dx$$

$$+ \Big/^{x_1} \int_{y_0}^{y_1} V_p\, \delta z\, dy - \Big/^{x_0} \int_{y_0}^{y_1} V_p\, \delta z\, dy = 0. \tag{9}$$

Now we are evidently at liberty to vary $z$ for any portion of the contour we please, leaving it unvaried for the remainder; and the four terms in (9) are therefore also completely independent; so that these terms must also be equated severally to zero, giving us the equations

$$\left.\begin{aligned} &\int_{x_0}^{x_1} \Big/^{y_1} k\, \delta z\, dx = 0, \qquad &&\int_{x_0}^{x_1} \Big/^{y_0} k\, \delta z\, dx = 0, \\ &\Big/^{x_1} \int_{y_0}^{y_1} V_p\, \delta z\, dy = 0, \qquad &&\Big/^{x_0} \int_{y_0}^{y_1} V_p \delta z\, dy = 0; \end{aligned}\right\} \tag{10}$$

where the first two integrals extend respectively along the two portions of the contour whose equations are $y = y_1$ and $y = y_0$; while the last two, if they exist at all, extend along the right lines whose equations are $x = x_1$ and $x = x_0$.

But $\delta z$ along any one of these four portions of the contour is entirely in our power, while its coefficient is not. It will therefore appear that we can only make the integral certainly vanish by supposing the coefficient of $\delta z$ to vanish throughout the whole range of the integration. We must therefore have

$$\Big/^{y_1} k = 0, \qquad \Big/^{y_0} k = 0, \qquad \Big/^{x_1} V_p = 0, \qquad \Big/^{x_0} V_p = 0; \tag{11}$$

where the substitutions merely indicate to what part of the contour the condition belongs.

**390.** Let us now consider what equations (11) imply.

The first two equations merely show that $k$ must vanish

along every portion of the contour for which $x$ is a variable, while the last two show that $V_p$ must vanish along both portions of the contour whose abscissæ are the constants $x_1$ and $x_0$, if such portions exist. Now restoring the value of $k$, and clearing fractions, we have along the first two portions of the contour

$$V_q\,dx - V_p\,dy = 0. \tag{12}$$

Moreover, this equation holds also along the other two portions of the contour, when such portions exist, and they furnish no other condition. For along either of these portions $dx$ vanishes, while $dy$, being taken along the right line $x = x_0$ or $x = x_1$, does not vanish, so that it is easy to see that the application of (12) to either of these portions would lead necessarily to $V_p = 0$.

It appears, then, that equation (12) must hold for the entire contour, and that there are no other equations, although we have already seen that this equation may represent more than one condition.

**391.** But before entering upon any further discussion, let us apply the results which we have obtained to Probs. LVII. and LVIII., beginning with the former. Here (12) gives at once

$$q\,dx - p\,dy = 0, \tag{13}$$

an equation which indicates that the required surface must at every point of its contour meet at right angles the limiting cylindrical walls. Now, theoretically speaking, if we could obtain the general integral of equation (10), Prob. LVII., involving two arbitrary functions, these functions might be determined so as to satisfy two conditions at the limits. But if the limiting walls should consist of a number of sides, curvilinear or rectilinear, it is evident that the application of (13) to each of these sides might involve as many distinct conditions as there are sides; so that we would expect in general

to find it impossible to form a surface which would satisfy the fundamental equation (10), and be also at right angles to more than two sides of a limiting wall; although it might happen that the surface which would satisfy two of these conditions would satisfy the others also.

If, as is usual, we suppose the *projected contour* to be some closed curve, the limiting wall has but two parts, those for which $y = y_1$ and $y = y_0$, and all the terms in $\delta U$ affected by the substitution of $x_1$ or $x_0$ disappear. In this case, therefore, we would infer that all the conditions of the question could be satisfied.

**392.** Let us now turn to Prob. LVIII. Here we find that equation (12) will give the condition

$$v^{(m-1)}(y\,dx - x\,dy) = 0. \tag{14}$$

Now as the second factor of this equation relates solely to the *projected contour*, it can become zero only when the portion of this contour along which it vanishes is a right line passing through the origin. But along any portion of the contour whose projection is not such a right line we must have $v = 0$. But along any portion of the *projected contour* which is not a right line passing through the origin, $\frac{y}{x}$ must be variable. Hence, in order to satisfy equation (5), Art. 372, $f\left(\frac{y}{x}\right)$ and consequently $F\left(\frac{y}{x}\right)$ must vanish throughout the entire surface, and equation (7), Art. 372, will become $z = xf'\left(\frac{y}{x}\right)$, which is the general equation of a conic surface, having its summit at the origin. See De Morgan's Diff. and Intg. Cal., p. 400, where $m$, $n$ and $p$ are to be made zero.

Now this conic surface must meet every element of the cylindrical wall, and the function $f'$ must be such as to enable

the surface to satisfy this condition. But it must appear upon reflection that this condition will not enable us to determine the form of $f'$, since it is evident that the conic surface might be of various characters and still touch every element of the cylindrical walls.

### Problem LX.

**393.** *Suppose we next demand that in Probs. LVII. and LVIII. the contour of the required surface shall always rest upon one or more given surfaces.*

This case is evidently analogous to that in which we are to connect two given curves by a curve having a certain maximum or minimum property, and we can in a similar manner pass from the primitive to the derived surface by first ascribing such variations to $z, p$ and $q$ as will give us a derived surface of any required form, and then so altering the dimensions of this surface as to cause it to intercept the bounding surface or surfaces.

This change of dimension will involve an alteration in the form of the *projected contour*, and to consider this contour in the most general manner, we shall, as before, suppose that it terminates in the lines $x = x_0$ and $x = x_1$, as we can then easily make the formulæ thus obtained applicable to any other case by reducing one or both these lines to points. We shall, moreover, for convenience, denote the four portions of either contour corresponding respectively to $y = y_0$, $y = y_1$, $x = x_0$ and $x = x_1$ by the terms *lower*, *upper*, *left* and *right*.

But in changing the form of the projected contour we need not vary the general values of $x$ or $y$, but merely those of $y_0$, $y_1$, $x_0$ and $x_1$. For we must remember that this contour encloses a certain plane surface, and that the general values of $x$ and $y$, as used in the double integral, must include the co-ordinates of every point of this plane surface; so that for every value of $x$ there are an infinity of values for $y$. If, there-

fore, we regard $x$ and $y$ as varying throughout the integral, as we evidently may, we in effect suppose the points of this plane to change their positions. But this is manifestly useless, it being sufficient to add to the ordinates $y_0$ and $y_1$, and to the abscissæ $x_0$ and $x_1$, infinitesimal increments $Dy_0$ $Dy_1$, $Dx_0$ and $Dx_1$, these quantities being independent of either sign, and representing the same increments as would be denoted by $\delta y$, and $dx_0$ and $dx_1$, if we had been obliged to vary in the same manner the form of the *projected contour* in the case of a single integral, the limits being also variable.

**394.** Let us now consider in detail the mode of obtaining $\delta U$, where $U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} V dy\, dx$, $V$ being any function of $x, y, z, p$ and $q$, the limiting values of $x$ and $y$ being also subject to variation.

First, varying $z, p$ and $q$ only, we have, to the second order,

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\{V_z\delta z + V_p\delta p + V_q\delta q\}dy\,dx$$

$$+\frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\{V_{zz}\delta z^2 + 2V_{zp}\delta z\,\delta p + V_{pp}\delta p^2 + 2V_{zq}\delta z\,\delta q$$

$$+ 2V_{pq}\delta p\,\delta q + V_{qq}\delta q^2\}dy\,dx. \quad (1)$$

This gives to the second order the change which $U$ will undergo when we pass from any primitive to any derived surface, the form of the projected contour or of the bounding walls remaining unaltered.

In the second place, let us consider the change which $U$ will undergo when we alter the dimensions of this derived surface in any infinitesimal manner we please, supposing $x_0$ and $x_1$ to remain unchanged. Since $U$ consists of the sum of the elements $dx\int_{y_0}^{y_1} V dy$, in which the integration is entirely independent of $x$, that quantity being regarded merely as a

constant, if it enter $V$ at all, the change sought will evidently be the sum of the additional changes which each element will undergo if, after having varied $z$, $p$ and $q$ only, we also vary $y_0$ and $y_1$ by adding any increment or decrement, $Dy_0$ and $Dy_1$.

Proceeding, then, with one of these elements as if it were the only integral in question, we shall obtain, in addition to the terms arising from the variation of $z$, $p$ and $q$ only, which are already included in (1), the terms

$$\int_{y_0}^{y_1} \left\{ VDy + \frac{1}{2} V_{\prime} Dy^2 + \delta V Dy \right\}, \tag{2}$$

where

$$V_{\prime} = V_y + V_z q + V_p s + V_q t \tag{3}$$

and

$$\delta V = V_z \delta z + V_p \delta p + V_q \delta q. \tag{4}$$

Hence, summing the changes in all the elements, we have

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \left\{ VDy + \frac{1}{2} V_{\prime} Dy^2 + \delta V Dy \right\} dx, \tag{5}$$

which gives the change sought, and must be added to (1).

**395.** We must now consider, in the third place, what change $U$ will undergo when we make any infinitesimal changes in the values of $x_0$ and $x_1$.

It is easy to see that we can, if we choose, pass from the primitive to the derived surface by first making the necessary changes in $y_0$ and $y_1$, or in the form of the upper or lower portions of the *projected contour*, $x_0$ and $x_1$ remaining fixed, as also the form of the surface; and then varying the surface under the supposition that $z$, $p$ and $q$ vary, and also that $x_0$ and $x_1$ become respectively $x_0 + Dx_0$ and $x_1 + Dx_1$, the new limiting values of $y$, which are $y_0 + Dy_0$ and $y_1 + Dy_1$, remaining fixed. Now the portion of $\delta U$ which will arise from varying

this surface, supposing $x_0$ and $x_1$ to remain fixed, is, as we have seen, found by taking the sum of equations (1) and (5), so that we have now to determine the portion which will arise merely from changing $x_0$ and $x_1$ into $x_0 + Dx_0$ and $x_1 + Dx_1$, and this added to (1) and (5) will evidently give the complete variation of $U$ to the second order.

**396.** Now by the changes in the limiting values of $y$ alone, $U$ will become

$$\int_{x_0}^{x_1}\int_{y_0+Dy_0}^{y_1+Dy_1} V dy\,dx \quad \text{or} \quad \int_{x_0}^{x_1} v\,dx, \tag{6}$$

where

$$v = \int_{y_0}^{y_1} Vdy + \Big/_{y_0}^{y_1}\{VDy + \text{etc.}\} = v^a + \Big/_{y_0}^{y_1}\{VDy + \text{etc.}\}. \tag{7}$$

Now since $v$ does not contain the limiting values of $x$, either explicitly or implicitly, any element $v\,dx$ will be independent of any changes in these limiting values, and therefore, although $v$ is a definite integral, we may employ the same reasoning as though it were not, and say that the change in $\int_{x_0}^{x_1} v\,dx$, due to the variation of the limits $x_0$ and $x_1$, must be

$$\Big/_{x_0}^{x_1}\left\{ v\,Dx + \frac{1}{2}\,v'\,Dx^2 + \delta v\,Dx \right\}. \tag{8}$$

Let us now approximate in (8) as far as the terms of the second order. We have

$$\Big/_{x_0}^{x_1} vDx = \Big/_{x_0}^{x_1}\left\{ v^a\,Dx + \Big/_{y_0}^{y_1} VDy\,Dx \right\}$$

$$= \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} Vdy\,Dx + \Big/_{x_0}^{x_1}\Big/_{y_0}^{y_1} VDy\,Dx, \tag{9}$$

where we must remember that the last term represents four terms involving merely the values of $V$, $Dy$ and $Dx$ at what

we may term the four corners of the surface, although $Dx_0$ and $Dx_1$ are infinitesimal and independent constants.

Now in reducing the second and third terms in (8) it will evidently be sufficient to regard $v$ as merely equal to $v^a$. Then, by equation (1), Art. 375, we have

$$\frac{dv^a}{dx} = \frac{d}{dx}\int_{y_0}^{y_1} V dy = \int_{y_0}^{y_1} \frac{dV}{dx} dy + \Big/_{y_0}^{y_1} V\frac{dy}{dx};$$

and therefore, to the second order,

$$\Big/_{x_0}^{x_1} \frac{1}{2}\frac{dv}{dx} Dx^2 = \Big/_{x_0}^{x_1} \frac{1}{2}\int_{y_0}^{y_1} \frac{dV}{dx} dy\, Dx^2 + \Big/_{x_0}^{x_1} \Big/_{y_0}^{y_1} \frac{1}{2} V\frac{dy}{dx} Dx^2, \quad (10)$$

$$\frac{dV}{dx} = V' = V_x + V_z p + V_p r + V_q s, \quad (11)$$

where accents as usual denote total differentials. We shall have also, to the second order,

$$\Big/_{x_0}^{x_1} \delta v\, Dx = \Big/_{x_0}^{x_1} \delta \int_{y_0}^{y_1} V dy\, Dx = \Big/_{x_0}^{x_1} \int_{y_0}^{y_1} \delta V dy\, Dx, \quad (12)$$

where $\delta V$ has the value given in (4). Hence, adding equations (9), (10) and (12), we obtain, for the last portion of $\delta U$,

$$\Big/_{x_0}^{x_1} \Big\{ \int_{y_0}^{y_1} V dy\, Dx + \Big/_{y_0}^{y_1} V Dy\, Dx + \frac{1}{2}\int_{y_0}^{y_1} V' dy\, Dx^2$$
$$+ \Big/_{y_0}^{y_1} \frac{1}{2} V \frac{dy}{dx} Dx^2 + \int_{y_0}^{y_1} \delta V dy\, Dx \Big\}. \quad (13)$$

**397.** Now by adding (1), (5) and (13), and then substituting the values of $V'$, $V$, and $\delta V$ from (3), (4) and (11), we shall have the complete variation of $U$ to the second order; and if $U$ is to be a maximum or a minimum, the terms of the first order must vanish, while those of the second must become invariably negative for a maximum and positive for a minimum.

As will be naturally surmised from their complicated nature, the determination of the sign of the terms of the second order transcends our present knowledge of variations, even when the form of $V$ is known; and we shall therefore in future consider only those terms in $\delta U$ which are of the first order.

Collecting these terms, we have

$$\delta U = \int_{x_0}^{x_1} \Big/_{y_0}^{y_1} V\,Dy\,dx + \Big/_{x_0}^{x_1} \int_{y_0}^{y_1} V\,Dx\,dy$$

$$+ \int_{x_0}^{x_1} \int_{y_0}^{y_1} \{V_z \delta z + V_p \delta p + V_q \delta q\}\,dy\,dx = 0. \tag{14}$$

Now transforming the double integral as in equation (4) of the preceding problem, we shall have, finally,

$$\delta U = \int_{x_0}^{x_1} \Big/_{y_0}^{y_1} \left\{ V_q - V_p \frac{dy}{dx} \right\} \delta z\,dx + \Big/_{x_0}^{x_1} \int_{y_0}^{y_1} V_p\,\delta z\,dy$$

$$+ \int_{x_0}^{x_1} \Big/_{y_0}^{y_1} V\,Dy\,dx + \Big/_{x_0}^{x_1} \int_{y_0}^{y_1} V\,Dx\,dy$$

$$+ \int_{x_0}^{x_1} \int_{y_0}^{y_1} \left\{ V_z - \frac{dV_p}{dx} - \frac{dV_q}{dy} \right\} \delta z\,dy\,dx = 0$$

$$= L + \int_{x_0}^{x_1} \int_{y_0}^{y_1} M\,\delta z\,dy\,dx. \tag{15}$$

**398.** Now it will appear, as in the preceding problem, that because the part of $\delta U$ under the sign of double integration cannot depend upon terms which relate to the limits only, these two parts must be independent, and that $L$ and $M$ must severally vanish. Therefore we see that here, as in single integrals, the differential equation from which the general solution must be obtained will be the same whatever may be the particular conditions which may be imposed at the limits.

Let us then examine the equation $L = 0$. It is easy to see

that if we can regard the quantities $\delta z$, $Dy$ and $Dx$ at all the limits as independent, the four terms in $L$ will be also independent, and we shall be obliged to equate them severally to zero. Hence, using $k$ as in the last problem, we must have

$$\left.\begin{aligned}\int_{x_0}^{x_1}\Big/_{y_0}^{y_1} k\,\delta z\,dx = 0, \qquad & \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V_p\delta z\,dy = 0,\\ \int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V\,Dy\,dx = 0, \qquad & \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V\,Dx\,dy = 0,\end{aligned}\right\} \tag{16}$$

in which the first two equations give the same conditions as in that problem.

Now in the third equation we must remember that $Dy$ is perfectly in our power for every point of the upper and lower portion of the *projected contour*, and is in fact what might be termed $\delta y$, if we had not agreed to suppose $x$ and $y$ incapable of receiving any variation; so that this integral will not certainly vanish unless we have $\Big/_{y_0}^{y_1} V = 0$.

In treating the fourth equation, we must remember that $Dx_0$ and $Dx_1$ do not, like $Dy_0$ and $Dy_1$, denote an infinity of quantities, but signify only one each, so that they are each arbitrary constants, and we must have $\Big/_{x_0}^{x_1}\int_{y_0}^{y_1} Vdy = 0$, and we cannot make any further reduction, because the integral is definite, and none of the quantities involved are in our power, although it is of course satisfied by $\Big/_{x_0}^{x_1} V = 0$.

We must then, in the present case, have the equations

$$k = 0, \qquad V = 0, \qquad V_p = 0, \qquad \int_{y_0}^{y_1} Vdy = 0; \tag{17}$$

the first two equations holding along the *upper* and *lower* contour, and the last two along the *right* and *left*. Or, as in the preceding problem, the condition $V_q\,dx - V_p\,dy = 0$ must hold for the entire contour; while we now add that the condition

$V = 0$ along the entire contour will satisfy all the remaining requirements of the limits, and will be necessary for all but the *right* and *left* portions of the contour, which might, perhaps, be satisfied by some other condition also.

**399.** But as it is necessary in the case of curves to impose some manner of restriction upon the extremities in order that $U$ may become a maximum or a minimum, so in the present case it is easy to see that the required surface cannot possess a maximum or a minimum property unless its contour be subjected to some sort of restriction.

Now the most general case which will arise is that of our problem—namely, where the required surface is to have its contour upon one or more given surfaces—and this case we will now proceed to consider.

**400.** Let the equation of any one of the limiting surfaces be of the form

$$dZ = PdX + QdY \quad \text{or} \quad Z = f(X, Y), \tag{18}$$

and let us first suppose it to be touched by a portion of the upper contour. Now if we pass a plane parallel to that of $yz$, at any distance $x$ from that plane, the sections cut from the required and the limiting surface will be two plane curves, which meet, and the equation of the curve cut from the limiting surface is $dZ = QdY$, while that of the other is $dz = qdy$. Therefore, so far as these two curves are concerned, we may regard $y$ as the independent variable, and $x$ as a constant, if it appear at all in their equations. Hence, when we change $y_1$ into $y_1 + Dy_1$, we may employ precisely the same reasoning as in Art. 69; so that, since $Q$ would replace $f'$ in that article, we shall, neglecting terms of the second order, have, as in equations (2), Art. 76,

$$\delta z_1 = (Q - q)_1 \, Dy_1,$$

and a similar equation will hold for the lower limit.

In like manner, for the limiting surface at the right, by passing a plane parallel to that of $xz$ at any distance $y$ from that plane, we find

$$\delta z_1 = (P - p)_1 Dx_1,$$

and a similar equation for the lower limit.

Or to render these equations more intelligible, we may write

$$\int_{y_0}^{y_1} \delta z = \int_{y_0}^{y_1} (Q - q)\, Dy, \qquad \int_{x_0}^{x_1} \delta z = \int_{x_0}^{x_1} (P - p)\, Dx; \quad (19)$$

or, to the second order, we shall have

$$\left.\begin{aligned} \int_{y_0}^{y_1} \delta z &= \int_{y_0}^{y_1} \left\{ (Q - q)\, Dy + \frac{1}{2}(T - t)\, Dy^2 - \delta q\, Dy \right\}, \\ \int_{x_0}^{x_1} \delta z &= \int_{x_0}^{x_1} \left\{ (P - p)\, Dx + \frac{1}{2}(R - r)\, Dx^2 - \delta p\, Dx \right\}. \end{aligned}\right\} \quad (20)$$

**401.** Now since equations (19) restrict the independence of $\delta z$ and $Dy$, and $\delta z$ and $Dx$ at both limits of $y$ and $x$, equations (17) will no longer hold true. But from (15) we may write

$$L = \int_{x_0}^{x_1} \int_{y_0}^{y_1} (k\delta z + V Dy) dx + \int_{x_0}^{x_1} \int_{y_0}^{y_1} (V_p \delta z + V Dx) dy = 0; (21)$$

and eliminating $\delta z$ by (19), we have

$$L = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \{ V + k(Q - q) \} Dy\, dx$$

$$+ \int_{x_0}^{x_1} \int_{y_0}^{y_1} \{ V + V_p(P - p) \} Dx\, dy = 0. \qquad (22)$$

Now it is evident that the quantities $Dy_0$, $Dy_1$, $Dx_0$ and $Dx_1$ are entirely independent of one another, as the fact that the contour is to be confined to certain surfaces in no way restricts

us in varying the form of the *projected contour.* Moreover, as before, $Dy$ is completely in our power for every point of the *upper* and *lower* contour, while, for either limit of $x$, $Dx$ is an arbitrary constant. Therefore, by the same reasoning as in the former case, (22) must give the equations

$$V + k(Q - q) = 0, \qquad \int_{y_0}^{y_1} \{V + V_p(P - p)\} dy = 0; \quad (23)$$

the first holding along either the *upper* or *lower* portion of the contour, and the second along either the *right* or *left.*

But $k = V_q - V_p y_x$; and also

$$dz = pdx + qdy \qquad \text{and} \qquad dZ = PdX + QdY,$$

the first being the equation of the required, and the second of any limiting surface; and since along their intersection $x, y, z$ and $X, Y, Z$ are identical, we must have along such intersection

$$pdx + qdy = Pdx + Qdy, \qquad \frac{dy}{dx} = -\frac{P - p}{Q - q}.$$

Substituting this value in $k$, and then the result in the first of equations (23), the conditions at the limits finally become

$$\left.\begin{aligned} V + V_p(P - p) + V_q(Q - q) = 0, \\ \int_{y_0}^{y_1} \{V + V_p(P - p)\} dy = 0. \end{aligned}\right\} \quad (24)$$

To discuss the terms of the second order we must employ equations (20) in the place of (19), proceeding as before, and setting aside all terms of that order which may arise. Then we shall have the same terms of the first order as before, while those which we have set aside must be added to the terms of the second order which we have already exhibited in equations (1), (5) and (13), thus rendering the complete

terms of that order still more complicated, and the determination of their sign a much more hopeless problem than before.

**402.** Now the first of equations (24) must hold along the entire *upper* and *lower* contour, and may represent as many distinct conditions as there are limiting surfaces touched by these portions of the contour. The second of these equations holds along the *right* and *left* contour only, and will be satisfied if we suppose the first to hold for these portions of the contour also, because along these portions, being parallel to the plane of $yz$, $q$ and $Q$ are equal, so that $Q - q$ will vanish.

The first condition, then, is necessary for the *upper* and *lower*, and will satisfy the requirements of the limits, should it hold throughout the entire contour, although the right and left portions may furnish some additional condition.

**403.** Let us now apply the foregoing theory to Probs. LVII. and LVIII., beginning with the former.

Here it is easy to see that equations (24) give the conditions

$$1 + Pp + Qq = 0, \qquad \int_{y_0}^{y_1} \frac{1 + Pp + q^2}{(1 + p^2 + q^2)^{\frac{1}{2}}} dy = 0. \qquad (25)$$

The first equation denotes that the required surface must meet at right angles all the limiting surfaces which are touched by its *upper* and *lower* contour, and the same condition might also prevail along the *right* and *left* portions, although we cannot assert that the second of equations (25) might not be satisfied in some other manner. In general, however, the *projected contour* will be a closed curve, in which case the *right* and *left* portion reduce to points, causing the second equation to disappear, and the first to hold along the entire contour.

As before, if we could obtain the general integral of equation (10), Prob. LVII., which would involve a number of arbi-

trary functions, not exceeding two, it would be necessary to determine these functions in such a manner as to satisfy equations (25).

**404.** Let us now turn to Prob. LVIII. Here equations (24) become

$$\left.\begin{array}{r} v^{m-1}\{v - mx(P-p) - my(Q-q)\} = 0, \\ \int_{y_0}^{y_1} v^{m-1}\{v - mx(P-p)\}dy = 0. \end{array}\right\} \qquad (26)$$

These equations will both be satisfied by $v = 0$ throughout the entire contour, which supposition would, as before, lead necessarily to a conic surface. Neglecting this supposition, we have

$$v - mx(P-p) - my(Q-q) = 0, \qquad v = z - px - qy.$$

Whence, substituting and transposing, we have

$$-m(Ppx + Pqy) + (m-1)(px+qy) = -z.$$

Adding $mz$ to both members and transposing, we have

$$m(z - Ppx - Pqy) = (m-1)(z - px - qy).$$

Whence

$$\frac{z - px - qy}{z - Ppx - Pqy} = \frac{m}{m-1}; \qquad (27)$$

which shows that if at any point of the upper or lower contour tangent planes be drawn, the first to the required and the second to the limiting surface, the portions of the axis of $z$ comprised between the origin and these planes respectively will be to each other as $m$ is to $m - 1$.

**405.** Having now reached the general discussion of the problem, let us consider more particularly the mode of deter-

mining the arbitrary functions in the various cases which may arise.

First suppose the contour to be a fixed boundary, and let it, for example, be a circle of radius $a$, having its centre on the axis of $z$, and its plane parallel to that of $xy$ at the distance $c$. Write $\frac{y}{x} = t$ and $x = \frac{[illegible]}{1-w}$. Then from the equation of the contour and from the general equation of the surface which now becomes

$$z = x^n F(t) - x^n f(t) = x^n F - x^n f, \qquad (29)$$

we have

$$\int^{b} x^2 - y^2 = a^2, \qquad \int^{b} \frac{\sqrt{x^2 - y^2}}{a} = 1, \qquad \int^{b} z = c,$$

$$x = \int^{b} \frac{a}{\sqrt{1-t^2}}, \qquad c = \int^{b} \frac{a^n F}{\sqrt{1-t^2}^{\,n}} - \int^{b} \frac{a^n f}{\sqrt{1-t^2}}. \qquad (30)$$

Having solved the last equation for $f$, we may then omit all signs of substitution, because the form of $f$ must remain the same for all values of $x$ and $y$ belonging to the required surface. Hence we have

$$f = \frac{\sqrt{1-t^2}}{a}\left(c - \frac{a^n F}{\sqrt{1-t^2}^{\,n}}\right). \qquad (31)$$

Now restoring the value of $t$, and substituting for $f$ in (29), we obtain

$$z = \frac{\sqrt{x^2-y^2}}{a}\left(c - \frac{a^n x^n F}{\sqrt{x^2-y^2}^{\,n}}\right) - x^n F. \qquad (32)$$

As the lower limiting values of $y$ furnish the same equations as the upper, we have no other condition by which to determine $F$, which may therefore be assumed arbitrarily.

Next suppose two circular arcs situated as before, having radii $a$ and $a'$, and that the given contour is to consist of a portion of the upper arc of each circle joined by any two curves whose projections on the plane of $xy$ shall be the right lines $x = x_0$ and $x = x_1$. Then $y_1$ and $y_0$ belong to these arcs only, and we obtain, as before,

$$\left.\begin{aligned} c &= \int^{y_1} \frac{a^n F}{\sqrt{(1+t^2)^n}} + \int^{y_1} \frac{af'}{\sqrt{1+t^2}}, \\ c &= \int^{y_0} \frac{a'^n F}{\sqrt{(1+t^2)^n}} + \int^{y_0} \frac{a'f'}{\sqrt{1+t^2}}. \end{aligned}\right\} \qquad (32)$$

Solving these equations for $f'$ and $F$, and omiting the signs of substitution for the same reason as before, we have

$$f' = (ca'^n - ca^n) \frac{\sqrt{1+t^2}}{aa'^n - a'a^n}, \qquad F = (ca - ca') \frac{\sqrt{(1+t^2)^n}}{aa'^n - a'a^n}. \quad (33)$$

Substituting these values in (28), we obtain

$$z(aa'^n - a'a^n) = (ca'^n - ca^n) \sqrt{x^2+y^2} + (ca - ca') \sqrt{(x^2+y^2)^n}. (34)$$

Thus we see that the two functions will be determined by the circumstance that the required surface is to pass through the two arcs, and we cannot impose any further conditions. Unless, therefore, the remaining portions of the fixed boundary be so assigned that they would lie necessarily upon this surface, the conditions of the problem cannot be all satisfied. We shall, however, have occasion to consider these functions again presently.

**406.** Let us next suppose that the required surface is to connect two planes whose equations are

$$z = ax + by + c \qquad \text{and} \qquad z = a'x + b'y + c'. \qquad (35)$$

From equation (5), Art. 372, observing that

$$f = \frac{m+2}{m-1}F \qquad \text{and} \qquad n = \frac{3}{1-m}, \tag{36}$$

we have $v$, or the numerator of (27), equals

$$\frac{m+2}{m-1}x^n F, \tag{37}$$

while the denominator of that equation must become either $c$ or $c'$. Hence (27) furnishes the conditions

$$\Big/^{v_1}(m+2)x^n F = mc, \qquad \Big/^{v_0}(m+2)x^n F = mc'. \tag{38}$$

We have also, along the upper contour,

$$z = (a+bt)x + c, \qquad z = x^n F + xf'. \tag{39}$$

Eliminating $z$ and $x^n F$ between these and the first of (38), we obtain

$$x = \frac{2c}{(m+2)(f'-a-bt)}; \tag{40}$$

and substituting this value in the first of equations (38), we have

$$(f'-a-bt)^n = \frac{2^n}{m}(m+2)^{1-n}c^{n-1}F; \tag{41}$$

and in like manner we find, along the lower contour,

$$(f'-a'-b't)^n = \frac{2^n}{m}(m+2)^{1-n}c'^{n-1}F. \tag{42}$$

If we solve (41) and (42) for $f'$ and $F$, and put $p$ for $\frac{n-1}{n}$, we shall obtain

$$\left.\begin{aligned} f' &= \frac{c'^p(a+bt) - c^p(a'+b't)}{c'^p - c^p}, \\ F &= m(m+2)^{n-1}\left\{\frac{a+bt-a'-b't}{2(c'^p - c^p)}\right\}^n. \end{aligned}\right\} \tag{43}$$

Although in (41) and (42) $t$ belonged respectively to the upper and lower *projected contour* only, in (43), for the reason already explained, it may belong to any point whatever of the required surface. Hence equation (28) becomes, after restoring the value of $t$,

$$z = \frac{c'^p(ax+by) - c^p(a'x+b'y)}{c'^p - c^p}$$

$$+ m(m+2)^{n-1}\left\{\frac{ax+by-a'x-b'y}{2(c'^p - c^p)}\right\}^n. \tag{44}$$

**407.** It will be remembered that, in the case of maximizing or minimizing any single integral $U$, it is necessary, in order to render the method of variations applicable, that no element of $U$ or of $\delta U$ shall become infinite within the range of the integration; and it will readily appear that when $U$ is a definite double integral the same principles will apply, since each element of $U$ is treated precisely as before. Now from (37) we have, in the present case,

$$U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} v^m\,dy\,dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1} C^m x^{nm} F^m\,dy\,dx, \tag{45}$$

where $C = \dfrac{m+2}{m-1}$. But $nm = \dfrac{3m}{1-m}$; so that it will appear, upon a little reflection, that $nm$ must be negative except when $m$ lies between zero and unity. Hence when $x = 0$, $x^{nm}$ must become infinite; and it will appear that to prevent $v$ from becoming certainly infinite, or at least indeterminate, we shall

be obliged to make $F$ vanish throughout $U$, which will give $z = xf'$, thus bringing back the conic surface, in which we must, as $v$ is zero when $x$ is zero, still reject all values of $m$ which would render $v^m$, $v^{m-1}$ or $v^{m-2}$ infinite, since the second and third of these quantities occur respectively as factors of the terms of the first and second orders.

In this case, there being but one function to determine, the first supposition in Art. 405 would determine the surface completely, requiring a right cone; so that

$$f' = \frac{c}{a}\sqrt{1 + \frac{y^2}{x^2}}, \qquad z = \frac{c}{a}\sqrt{x^2 + y^2}.$$

In Art. 392 $f'$ would, as we have seen, remain indeterminate, and indeed it is easy to see that we could have no finite minimum while the limiting values of $z$ remain variable. In this case equation (27) is inapplicable, since in obtaining it we assumed that $v$ did not vanish.

## Problem LXI.

**408.** *It is required to determine the form of the surface which will maximize or minimize the expression*

$$U = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \sqrt{p^2 + q^2}\, dy\, dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1} V dy\, dx. \qquad (1)$$

Here

$$V_z = 0, \qquad V_p = \frac{p}{\sqrt{p^2 + q^2}}, \qquad V_q = \frac{q}{\sqrt{p^2 + q^2}}; \qquad (2)$$

and observing that $p_{,} = q' = s$, the equation $M = 0$ will reduce to

$$q^2 r - 2pqs + p^2 t = 0. \qquad (3)$$

This equation may be integrated by the method of Monge;

and adopting the notation of De Morgan, page 719, we have, putting $z$ for $u$,

$$R = q^2, \qquad S = -2pq, \qquad T = p^2,$$

$$\alpha = q^2dy^2 + 2pq\,dy\,dx + p^2dx^2, \qquad V = 0,$$

$$\sigma = q^2dp\,dy + p^2dq\,dx, \qquad \mu = -\frac{p}{q}, \qquad dz = pdx + qdy.$$

Now if $\alpha$ vanish, we see from the last equation that $dz$ will vanish also, and *vice versa;* and by theory $\sigma$ will also vanish. Substituting $\mu\,dx$ for $dy$, the equation $\sigma = 0$ gives $qdp - pdq = 0$. Whence we may write $\frac{p}{q} = -f(z) = -f$. Again, when $\alpha = 0$, we have

$$dy - \mu\,dx \quad \text{or} \quad dy + \frac{p}{q}dx \quad \text{or} \quad dy - f\,dx = 0, \tag{4}$$

where we must remember that $f$ is to be regarded as a constant, because $dz$ is zero or $z$ is a constant. Hence

$$y - xf = F(z) = F.$$

The complete integral of (3) is, then,

$$y = xf(z) + F(z) = xf + F, \tag{5}$$

where $f$ and $F$ are any functions of $z$ whatever.

**409.** Let us first suppose the limiting values of $x$, $y$ and $z$ to be fixed, or that the surface is to pass through some fixed boundary, and let us require, as a particular case, that two portions of this boundary shall be given by the equations

$$/^{y_1}(x^2 + y^2) = a^2, \qquad /^{y_1}z = /^{y_1}\frac{my}{x}, \tag{6}$$

$$/^{y_0}(x^2 + y^2) = a'^2, \qquad /^{y_0}z = /^{y_0}\frac{m'y}{x}. \tag{7}$$

Then, for the upper limit, we have

$$y = \frac{zx}{m}, \qquad x = \frac{a}{\sqrt{1 + \frac{y^2}{x^2}}} = \frac{a}{\sqrt{1 + \frac{z^2}{m^2}}} = \frac{am}{\sqrt{m^2 + z^2}},$$

$$y = \frac{az}{\sqrt{m^2 + z^2}}.$$

Therefore, by (5), we have, for the upper limit,

$$\frac{az}{\sqrt{m^2 + z^2}} = \frac{amf}{\sqrt{m^2 + z^2}} + F. \tag{8}$$

Similarly we obtain, for the lower limit of $y$,

$$\frac{a'z}{\sqrt{m'^2 + z^2}} = \frac{a'm'f}{\sqrt{m'^2 + z^2}} + F. \tag{9}$$

Solving for $f$ and $F$, remembering that the results will no longer refer to the contour only, but will hold for every point of the required surface, we shall obtain

$$f = z\frac{a\sqrt{m'^2 + z^2} - a'\sqrt{m^2 + z^2}}{am\sqrt{m'^2 + z^2} - a'm'\sqrt{m^2 + z^2}},$$

$$F = \frac{aa'(m - m')z}{am\sqrt{m'^2 + z^2} - a'm'\sqrt{m^2 + z^2}}.$$

Now if the surface determined by the substitution of these values of $f$ and $F$ in (5) do not necessarily fulfil all the requirements of the problem regarding other portions of the fixed boundary, we conclude that these conditions cannot all be satisfied.

**410.** Next, suppose we give merely the limiting values of $x$ and $y$, those of $z$ remaining variable; that is, suppose we

give merely the form of the *projected contour*, or of the cylindrical walls. Then we see from (2) that equation (12), Art. 390, will furnish the condition

$$qdx - pdy = 0; \tag{10}$$

which shows that the required surface must meet these walls at right angles.

To discuss the form of the functions, let us suppose the wall to be a right circular cylinder, having the axis of $z$ as its axis. Then along the *projected contour* we have $dy = -\dfrac{xdx}{y}$, and (10) gives, by substitution,

$$px + qy = 0. \tag{11}$$

But by differentiating (5) with regard to $x$ and $y$ respectively, we find

$$px = \frac{-xf}{xf_z + F_z}, \qquad qy = \frac{y}{xf_z + F_z} = \frac{xf + F}{xf_z + F_z}.$$

Hence (11) gives $F = 0$; and (5) becomes $y = xf$, which may evidently be put under the form

$$z = f^{-1}\left(\frac{y}{x}\right) = f^{-1}. \tag{12}$$

The function $f^{-1}$ will remain undetermined unless we assume some other form for a portion of the cylindrical wall. Suppose, then, another portion to be elliptical, giving

$$dy = -\frac{bx\,dx}{ay}.$$

Then along it we have, as before,

$$bpx + aqy = 0. \tag{13}$$

Putting $t$ for $\frac{y}{x}$, we shall have from (12), which must now hold,

$$px = xf_t^{-1}t_x = -f_t^{-1}\frac{y}{x}, \quad qy = yf_t^{-1}t_y = f_t^{-1}\frac{y}{x}.$$

Whence (13) gives

$$(a-b)f_t^{-1} = 0, \quad f^{-1} = c, \quad z = c.$$

**411.** Let us next suppose the edges of the surface are required merely to rest upon one or more given surfaces. Then, substituting from (2) in equations (24), Art. 401, we find the conditions at the limits to be

$$Pp + Qq = 0, \quad \int_{y_0}^{y_1} \frac{q^2 + Pp}{\sqrt{(p^2+q^2)^2}}\,dy = 0. \tag{14}$$

The reader can readily apply in any particular case the first of these conditions to the determination of the arbitrary functions.

**412.** When the limiting values of $x$ and $y$ are fixed, whether those of $z$ be subject to variation or not, we find the terms of the second order to be

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{q^2\delta p^2 - 2pq\delta p\delta q + p^2\delta q^2}{(p^2+q^2)^{\frac{3}{2}}}\,dy\,dx$$

$$= \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{(q\delta p - p\delta q)^2}{(p^2+q^2)^{\frac{3}{2}}}\,dy\,dx. \tag{15}$$

Hence, since we suppose the denominator of (15) to be positive, we may conclude that $U$ will become a minimum for all solutions which do not give rise to infinite values for any element of $\delta U$; unless, indeed, it be possible to assign such values to $\delta p$ and $\delta q$ as will cause every element of (15) to vanish.

## Problem LXII.

**413.** *It is required to determine the form of the surface which will maximize or minimize the expression*

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}(z - px - qy)\,dy\,dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1} v\,dy\,dx, \qquad (1)$$

*while at the same time the variations of $p$ and $q$ are always to be so taken that the expression*

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\sqrt{p^2 + q^2}\,dy\,dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1} v'\,dy\,dx \qquad (2)$$

*may always have an assigned constant value.*

This is evidently a problem of relative maxima and minima, and we can treat it by Euler's method precisely as in the case of single integrals. For, supposing first the limiting values of $x$, $y$ and $z$ to be fixed, the reasoning of Bertrand, explained in Art. 93, which the reader is supposed to re-peruse, can, in the following manner, be extended to this problem.

Since the terms at the limits vanish, we must have

$$\left.\begin{aligned} \int_{x_0}^{x_1}\int_{y_0}^{y_1}\delta v\,dy\,dx \quad \text{or} \quad \int_{x_0}^{x_1}\int_{y_0}^{y_1} V\delta z\,dy\,dx = 0, \\ \int_{x_0}^{x_1}\int_{y_0}^{y_1}\delta v'\,dy\,dx \quad \text{or} \quad \int_{x_0}^{x_1}\int_{y_0}^{y_1} V'\delta z\,dy\,dx = 0; \end{aligned}\right\} \qquad (3)$$

where

$$V = v_z - \frac{dv_p}{dx} - \frac{dv_q}{dy}, \qquad V' = v_z' - \frac{dv_p'}{dx} - \frac{dv_q'}{dy}. \qquad (4)$$

Now suppose the required surface to have been obtained, and on it select any two portions in such a manner that for every point of either portion, when that portion is considered separately, both $V$ and $V'$ may preserve an invariable sign. Then

vary $z$ throughout these portions only, leaving the remainder of the surface unvaried in form. Also make the sign of $\delta z$ invariable throughout each, giving to it in the two portions like signs when those of $V$ are unlike, and *vice versa.*

In this way, by giving suitable values to $\delta z$, we can, as in Art. 93, satisfy the first of equations (3). But the second of these equations may be written

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\delta v'\,dy\,dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1} f\,V\,\delta z\,dy\,dx, \qquad f = \frac{V'}{V}; \tag{5}$$

the variations of $z$ being taken as before; so that unless $f$ be a constant, we can certainly effect that the double integrals taken throughout the two portions shall be numerically unequal, and hence the second of equations (3) would not be satisfied.

The remaining reasoning, by which the necessity of Euler's method is established, is precisely like that of Art. 93.

If the limiting values of $x$, $y$ and $z$ are also subject to variation, the method of Euler is still equally applicable. For suppose the required surface were to be bounded by certain cylindrical walls or by certain surfaces. Then, since we are not compelled to vary the limiting values of $x$, $y$ or $z$, the required surface must evidently be of that kind which will satisfy all the conditions of the problem when the contour is to be fixed, the only question being to determine the conditions which must hold along the contour; and since, in double as in single integrals, the fundamental equation obtained in discussing any problem of absolute maxima or minima is the same whatever be the conditions which are to hold at the limits, the applicability of Euler's method is apparent, as in Art. 96.

**414.** We see, then, that we are in the present case to discuss the conditions which will maximize or minimize absolutely the expression

$$U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\left(z - px - qy + a\sqrt{p^2+q^2}\right) dy\, dx$$

$$= \int_{x_0}^{x_1}\int_{y_0}^{y_1} V dy\, dx. \tag{1}$$

Here

$$V_z = 1, \qquad V_p = -x + \frac{ap}{\sqrt{p^2+q^2}}, \qquad V_q = -y + \frac{aq}{\sqrt{p^2+q^2}}; \tag{2}$$

so that, writing $b = -\dfrac{3}{a}$, the equation $M = 0$ gives

$$q^2 r - 2pqs + p^2 t = b(p^2+q^2)^{\frac{3}{2}}. \tag{3}$$

This equation is integrable by the method of Monge. See Boole's Diff. Eqs, Chapter XV., or De Morgan's Diff. and Integ. Calc., page 719. Adopting the notation of the latter, we may write

$$R = \frac{q^2}{(p^2+q^2)^{\frac{3}{2}}}, \quad S = -\frac{2pq}{(p^2+q^2)^{\frac{3}{2}}}, \quad T = \frac{p^2}{(p^2+q^2)^{\frac{3}{2}}}, \quad V = b,$$

$$\alpha = \frac{q^2 dy^2 + 2pq\, dx\, dy + p^2 dx^2}{(p^2+q^2)^{\frac{3}{2}}}, \qquad \mu = -\frac{p}{q},$$

$$\sigma = \frac{q^2 dp\, dy + p^2 dq\, dx}{(p^2+q^2)^{\frac{3}{2}}} - b\, dx\, dy,$$

$$\tau = \frac{q^2(dq\, dy - dp\, dx) + 2pq\, dq\, dx}{(p^2+q^2)^{\frac{3}{2}}} + b\, dx^2,$$

$$dz = p\, dx + q\, dy.$$

Now the condition $\alpha = 0$ renders $dz$ zero, and also gives $dy - \mu dx = 0$; so that we may write

$$\int \frac{p}{q} dx = -y + f_1(z) = -y + f_1. \tag{A}$$

Now substituting $\mu dx$ for $dy$ in $\sigma$ and $\tau$, which must also become zero when $\alpha$ is zero, we obtain

$$\frac{p^2dq - pqdp}{(p^2+q^2)^{\frac{3}{2}}} + b\frac{p}{q}dx = 0, \qquad \frac{q^2dp - pqdq}{(p^2+q^2)^{\frac{3}{2}}} + bdx = 0.$$

Integrating these equations, we obtain

$$\frac{q}{\sqrt{p^2+q^2}} = -b\int\frac{p}{q}dx + f_1(z), \quad \frac{p}{\sqrt{p^2+q^2}} = -bx + f_2(z). \qquad \text{(B)}$$

Now by squaring and adding equations (B), and substituting from (A) the value of $\int\frac{p}{q}dx$, we shall obtain the integral sought.

The complete integral of this equation is, therefore,

$$\frac{1}{b^2} = (x+f(z))^2 + (y+F(z))^2 = (x+f)^2 + (y+F)^2. \qquad (4)$$

**415.** This equation is easily interpreted. For suppose a circle whose radius is $\frac{1}{b}$; and while keeping its plane always parallel to that of $xz$, let its centre move along some curve in space whose equations shall be

$$X = -f, \qquad Y = -F, \qquad Z = z. \qquad (5)$$

Then it will readily appear that (4) represents the equation of the surface generated by the circumference of this circle as it moves along the given curve, and that when we shall in any particular case have determined the form of the two arbitrary functions, $f$ and $F$, we shall know the nature of the curved directrix of this surface. When the contour of the surface is fixed, the functions must be determined in accordance with this condition.

If the bounding walls only are to have a given form, equation (12), Art. 390, will give

$$(y\sqrt{p^2+q^2}-aq)dx = (x\sqrt{p^2+q^2}-ap)dy, \tag{6}$$

and $f$ and $F$ must be determined so as to satisfy this equation.

When the required surface is to be limited by one or more given surfaces, the first of equations (24), Art. 401, which is the only one of importance, will become, by substituting from (2),

$$z - Px - Qy + a\frac{Pp+Qq}{\sqrt{p^2+q^2}} = 0, \tag{7}$$

and $f$ and $F$ must then be determined in accordance with this condition.

**416.** Of these cases we will consider but one—that in which the required surface is to be limited by two planes, each passing through the origin, and having for their equations

$$z = cx + c_1y, \qquad z = c'x + c_1'y. \tag{8}$$

In this case (7) will give

$$/^{v_1}(cp+c_1q) = 0, \qquad /^{v_0}(c'p+c_1'q) = 0. \tag{9}$$

But from (4) we obtain

$$\left.\begin{aligned} p &= -\frac{x+f}{(x+f)f_z+(y+F)F_z}, \\ q &= -\frac{y+F}{(x+f)f_z+(y+F)F_z}. \end{aligned}\right\} \tag{10}$$

Hence, by the use of (8), equation (9) gives

$$/^{v_1}(z+cf+c_1F) = 0, \qquad /^{v_0}(z+c'f+c_1'F) = 0. \tag{11}$$

From (5) these conditions may be written

$$Z = cX + c_1 Y, \qquad Z = c'X + c_1' Y. \tag{12}$$

From these equations it at once appears that if there were but one limiting plane, the centre of the generating circle would be compelled to remain always in that plane, and that in the present case the centre must move along the intersection of the two limiting planes. This will give an oblique cylinder having a circular base, the line in which the two planes intersect being its axis. We can, of course, determine $f$ and $F$ in the usual way, thus obtaining, from (11),

$$f = \frac{c_1' - c_1}{cc_1' - c'c_1} z, \qquad F = \frac{c - c'}{cc_1' - c'c_1} z.$$

Moreover, when in any particular case we have determined the functions $f$ and $F$, we shall then be able to determine also the constant $b$ or $-\frac{3}{a}$. For, as in the case of single integrals, we have the condition that one of the double integrals is to remain constant, and we may suppose a definite value to have been assigned to it.

**417.** In considering the terms of the second order the same reasoning will hold as in the case of single integrals. For the variations of $z$, $p$ and $q$ are subject to a certain restriction which we cannot explicitly express, and the method of Euler will cause the terms of the first order to vanish whether these variations are restricted or not. But the variations are still restricted, and when we come to the terms of the second order it is conceivable that even when they do not indicate a maximum or a minimum, the variations being unrestricted, they would do so if we could employ such variations only as would permit one of the double integrals to remain always constant, which, however, we have no means of doing. But when these terms indicate an absolute maximum or minimum

—that is, for all systems of variations—there would seem to be no doubt as to the existence of a relative maximum or minimum also.

In the present problem, when the limiting values of $x$ and $y$ are fixed, the terms of the second order are the same as in equation (15), Art. 412, only multiplied by $a$. Hence we may in this case conclude that $U$ will be a maximum or a minimum according as $a$ is negative or positive.

## Problem LXIII.

**418.** *It is required to determine the form which a surface of given area whose edges are in some manner confined must assume in order that the depth of its centre of gravity may be a maximum.*

The given area is

$$A = \int_{x_0}^{x_1}\int_{y_0}^{y_1} \sqrt{1+p^2+q^2}\,dy\,dx;$$

and assuming the axis of $z$ vertically downward, we have, for the depth of the centre of gravity,

$$D = \frac{1}{A}\int_{x_0}^{x_1}\int_{y_0}^{y_1} z\sqrt{1+p^2+q^2}\,dy\,dx,$$

which is to be a maximum. Or, since $A$ is to be a constant, we may say that $\int_{x_0}^{x_1}\int_{y_0}^{y_1} z\sqrt{1+p^2+q^2}\,dy\,dx$ is to be a maximum, while $\int_{x_0}^{x_1}\int_{y_0}^{y_1}\sqrt{1+p^2+q^2}\,dy\,dx$ is to remain constant. Hence, employing Euler's method, we may write

$$U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}(z-a)\sqrt{1+p^2+q^2}\,dy\,dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1} V\,dy\,dx. \quad (1)$$

Here
$$\left.\begin{array}{c} V_z = \sqrt{1+p^2+q^2}, \\ V_p = \dfrac{(z-a)p}{\sqrt{1+p^2+q^2}}, \qquad V_q = \dfrac{(z-a)q}{\sqrt{1+p^2+q^2}}, \end{array}\right\} \tag{2}$$

Hence the equation $M = 0$ reduces to

$$1+p^2+q^2 = (z-a)\{(1+q^2)r - 2pqs + (1+p^2)t\}. \tag{3}$$

This equation is not integrable; but calling $R$ and $R'$ the principal radii of curvature, and estimating the signs properly, (3) may be written

$$\frac{1}{R} + \frac{1}{R'} = \frac{1}{(z-a)\sqrt{1+p^2+q^2}}, \tag{4}$$

because

$$\frac{1}{R} + \frac{1}{R'} = \frac{(1+q^2)r - 2pqs + (1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}}.$$

Equation (4) shows that the mean radius of curvature of the required surface at any point is twice the normal extended until it meets the plane whose equation is $z = a$. The same equation also indicates an analogy between this surface and the catenary, which gives, as we have already seen, the solution for a similar problem relative to plane curves. (See Art 282.)

If the contour, instead of passing through some fixed curve, be confined to certain cylindrical walls only, we must have, from equation (12), Art. 390, $q\,dx - p\,dy = 0$, showing that the surface sought must meet these walls at right angles.

When the edges of the required surface must be upon one or more given surfaces, the equation of any one of which is $dZ = PdX + QdY$, the first of equations (24), Art. 401, will give the condition $1 + Pp + Qq = 0$, showing that the required surface must be normal to the limiting surfaces.

## Problem LXIV.

**419.** *It is required to determine the form of the surface whose area shall be a minimum, and which shall cover a given volume on a horizontal plane.*

Here, since the given volume is $\int_{x_0}^{x_1}\int_{y_0}^{y_1} z\,dy\,dx$, we may write at once

$$U=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\left(a\sqrt{1+p^2+q^2}-z\right)dy\,dx=\int_{x_0}^{x_1}\int_{y_0}^{y_1}V\,dy\,dx. \qquad (1)$$

Here

$$V_z=-1, \qquad V_p=\frac{ap}{\sqrt{1+p^2+q^2}}, \qquad V_q=\frac{aq}{\sqrt{1+p^2+q^2}}; \qquad (2)$$

and the equation $M=0$ will give

$$\frac{(1+q^2)r-2pqs+(1+p^2)t}{\sqrt{(1+p^2+q^2)^3}}+\frac{1}{a}=0. \qquad (3)$$

This equation, which is not integrable, gives, as in the preceding problem, by a contrary estimation of signs,

$$\frac{1}{R}+\frac{1}{R'}=\frac{1}{a}. \qquad (4)$$

Hence the required surface must be such that its mean curvature at every point may be constant.

**420.** We already know that it will be necessary to the existence of a maximum or a minimum that the contour shall either be fixed or rest upon some surface or surfaces, the calculus of variations affording in the first case no further equations; and we are unable to integrate (3). But when, in the

second case, these limiting surfaces are certain cylindrical walls normal to the plane of $xy$, equation (12), Art. 390, gives

$$q\,dx - p\,dy = 0, \tag{5}$$

the meaning of which we know.

When, however, the limiting surfaces to which the contour is to be confined may have any given form, the first of equations (24), Art. 401, gives

$$z\sqrt{1+p^2+q^2} - a(1 + Pp + Qq) = 0. \tag{6}$$

Suppose, for example, the limiting surface to be a plane whose equation is $z = h$. Then (6) will give

$$\frac{1}{\sqrt{1+p^2+q^2}} = \frac{h}{a}. \tag{7}$$

Hence the angle $A$ which the tangent plane to the required surface at any point of the contour makes with the plane of $xy$ must be a constant, since the first member of (7) is $\frac{1}{\sec A}$ or $\cos A$.

When $h = 0$, we must have $\cos A = 0$, and the required surface meets the plane of $xy$, and is normal to it. The surface of a hemisphere of radius $2a$ would evidently, in this case, satisfy all the conditions of the question so far as the terms of the first order are concerned; but a satisfactory investigation of those of the second order would probably be impossible.

When the limiting values of $x$ and $y$ are fixed, the terms of the second order may be written

$$\delta U = \frac{a}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{(q\delta p - p\delta q)^2}{(1+p^2+q^2)^{\frac{3}{2}}}\,dy\,dx, \tag{8}$$

and, as in the case of a spherical surface, the radius is $2a$, and is positive, we may conclude that if we vary the form of the surface only, the circular base remaining unvaried, the surface will be a minimum.

**421.** To give a more comprehensive view of the method of M. Sarrus in the treatment of double integrals, we now proceed to a more general problem. But the reader who desires may omit the discussion of the following example.

### Problem LXV.

*It is required to maximize or minimize the expression* $U=\int_{x_0}^{x_1}\int_{y_0}^{y_1} V dy\, dx$, *where* $V$ *is any function of* $x, y, z, p, q, r, s$, *and* $t$.

It is evident that, supposing the limiting values of $x$ and $y$ to be also variable, we shall have

$$\delta U=\int_{x_0}^{x_1}\Big/_{y_0}^{y_1} VDy\, dx+\Big/_{x_0}^{x_1}\int_{y_0}^{y_1} VDx\, dy$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\{V_z\delta z+V_p\delta p+V_q\delta q$$

$$+V_r\delta r+V_s\delta s+V_t\delta t\}dy\, dx=0. \quad (1)$$

Now all the terms except the last three are to be transformed and arranged as in equation (15), Art. 397, so that we have to consider these three terms only.

**422.** By equation (3), Art. 377, we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_r\frac{d\delta p}{dx}\, dy\, dx=$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_r'\delta p\, dy\, dx+\Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V_r\delta p\, dy-\int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_r\delta p\, y_x dx. \quad (2)$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_r'\frac{d\delta z}{dx}\, dy\, dx=$$

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_r''\delta z\, dy\, dx-\Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V_r'\delta z\, dy+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_r'\delta z\, y_x dx. \quad (3)$$

Moreover, by equation (4), Art. 378, we have

$$-\int_{x_0}^{x_1}\Big/^{y_1} y_x V_r \frac{d\delta z}{dx}\,dx = \int_{x_0}^{x_1}\Big/^{y_1}(y_x V_r)'\delta z\,dx - \Big/_{x_0}^{x_1}\Big/^{y_1} y_x V_r \delta z$$

$$+\int_{x_0}^{x_1}\Big/^{y_1}(y_x)^2 V_r\,\delta q\,dx + \int_{x_0}^{x_1}\Big/^{y_1} y_x (y_x V_r)_{,}\delta z\,dx. \qquad (4)$$

$$\int_{x_0}^{x_1}\Big/^{y_0} y_x V_r \frac{d\delta z}{dx}\,dx = -\int_{x_0}^{x_1}\Big/^{y_0}(y_x V_r)'\delta z\,dx + \Big/_{x_0}^{x_1}\Big/^{y_0} y_x V_r \delta z$$

$$-\int_{x_0}^{x_1}\Big/^{y_0}(y_x)^2 V_r\,\delta q\,dx - \int_{x_0}^{x_1}\Big/^{y_0} y_x (y_x V_r)_{,}\delta z\,dx. \qquad (5)$$

Now we must observe that every $y_x$ refers to the contour only, and hence it varies with $x$, but is independent of the general values of $y$. Hence

$$(V_r y_x)' = V_r y_{xx} + V_r' y_x, \qquad (V_r y_x)_{,} = y_x V_{r,}. \qquad (6)$$

Substituting these values and collecting results, we may write

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_r \delta r\,dy\,dx = \int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\{V_r y_{xx} + 2V_r' y_x + V_{r,}(y_x)^2\}\delta z\,dx$$

$$+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_r (y_x)^2\,\delta q\,dx - \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V_r'\delta z\,dy + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V_r \delta p\,dy$$

$$-\Big/_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_r y_x\,\delta z + \int_{x_0}^{x_1}\int_{y_0}^{y_1} V_r''\delta z\,dy\,dx. \qquad (7)$$

Again, by equation (3), Art. 377, we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_s \frac{d\delta q}{dx}\,dy\,dx =$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_s'\delta q\,dy\,dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V_s \delta q\,dy - \int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_s y_x \delta q\,dx. \quad (8)$$

By equation (5), Art. 379, we have

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_s' \frac{d\delta z}{dy}\, dy\, dx =$$

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_{s,}'\delta z\, dy\, dx - \int_{x_0}^{x_1} \Big/_{y_0}^{y_1} V_s'\delta z\, dx; \tag{9}$$

and by equation (6), Art. 380, we have

$$\left.\begin{aligned} \Big/^{x_1}\int_{y_0}^{y_1} V_s \frac{d\delta z}{dy}\, dy &= -\Big/^{x_1}\int_{y_0}^{y_1} V_{s,}\delta z\, dy + \Big/^{x_1}\Big/_{y_0}^{y_1} V_s\delta z, \\ -\Big/^{x_0}\int_{y_0}^{y_1} V_s \frac{d\delta z}{dy}\, dy &= \Big/^{x_0}\int_{y_0}^{y_1} V_{s,}\delta z\, dy - \Big/^{x_0}\Big/_{y_0}^{y_1} V_s\delta z. \end{aligned}\right\} \tag{10}$$

Hence, collecting results, we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_s\delta s\, dy\, dx = -\int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_s'\delta z\, dx - \int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_s y_x\delta q\, dx$$

$$-\Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V_{s,}\delta z\, dy + \Big/_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_s\delta z + \int_{x_0}^{x_1}\int_{y_0}^{y_1} V_{s,}'\delta z\, dy\, dx. \tag{11}$$

Lastly, by equation (5), Art. 379, we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_t \frac{d\delta q}{dy}\, dy\, dx = -\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_{t,}\delta q\, dy\, dx + \int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_t\delta q\, dx,$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_{t,} \frac{d\delta z}{dy}\, dy\, dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1} V_{t,,}\delta z\, dy\, dx - \int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_{t,}\delta z\, dx.$$

Whence

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} V_t\delta t\, dy\, dx =$$

$$-\int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_{t,}\delta z\, dx + \int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V_t\delta q\, dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1} V_{t,,}\delta z\, dy\, dx. \tag{12}$$

Adding these results, and also the second member of equation (15), Art. 397, we finally obtain

$$\delta U =$$

$$\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\{V_q - V_p y_x + V_r y_{xx} + 2V_r' y_x + V_{r_\prime}(y_x)^2 - V_s' - V_{t_\prime}\}\delta z\, dx$$

$$+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\{V_r(y_x)^2 - V_s y_x + V_t\}\delta q\, dx$$

$$+\Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\{V_p - V_r' - V_{s_\prime}\}\delta z\, dy$$

$$+\Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V_r \delta p\, dy + \Big/_{x_0}^{x_1}\Big/_{y_0}^{y_1}\{V_s - V_r y_x\}\delta z$$

$$+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V\, Dy\, dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V\, Dx\, dy$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\{V_z - V_p' - V_{q_\prime} + V_r'' + V_{s_\prime}' + V_{t_{\prime\prime}}\}\delta z\, dy\, dx = 0. \quad (13)$$

**423.** Now when $U$ is to be a maximum or minimum, we must, as before, have $M = 0$ irrespectively of the conditions which are to hold at the limits. This equation, which must subsist for every point of the required surface, will be in general of the fourth order, and its solution, when any exists, will not contain more than four arbitrary functions.

Next, if in the terms at the limits we regard the quantities $\delta z$, $\delta p$, $\delta q$, $Dy$ and $Dx$ as independent at each limit, we shall evidently obtain the following system of equations:

$$\left.\begin{aligned} V_r y_{xx} + V_{r_\prime}(y_x)^2 + (2V_r' - V_p)y_x + V_q - V_s' - V_{t_\prime} = 0, \\ V_r(y_x)^2 - V_s y_x + V_t = 0, \end{aligned}\right\} \quad (14)$$

$$V_p - V_r' - V_{s_\prime} = 0, \qquad V_r = 0, \quad (15)$$

$$V_s - V_r y_x = 0, \quad (16)$$

$$V = 0, \qquad \int_{y_0}^{y_1} V dy = 0; \quad (17)$$

where (14) and the first of (17) hold along the *upper* and *lower* contour, (15) and the second of (17) along the *left* and *right* portions when they exist, while (16) holds only for the four corners, or the junction of the different portions of the contour, the differentials $y_x$, etc., at these points being taken with reference to that one of the two intersecting portions which we may happen to be considering.

But under the present supposition the total number of equations at the limits would be (16), whereas we have at the most not more than four arbitrary functions with which to satisfy them; so that, as before, we must impose some restriction upon the contour which will reduce the number of these equations.

**424.** If we suppose the form of the *projected contour* to be fixed, equations (17) will disappear, and we shall have but twelve equations at the limits; and if, in addition, we suppose the *left* and *right* portions to be wanting, equations (15) and (16) will also cease to exist, and we shall have but four equations at the limits. In this case, therefore, in which the *projected contour* consists merely of two curves which meet, we may reasonably suppose that a complete solution might be possible.

We can render equations (14) somewhat more symmetrical. For differentiating the second, regarding $y$ as a function of $x$, as indeed it is along the *projected contour*, we obtain

$$(2V_r y_x - V_s)y_{xx} + V_{r_,}(y_x)^3 + (V_r' - V_{s_,})(y_x)^2$$
$$+ (V_{t_,} - V_s')y_x + V_t' = 0, \quad (18)$$

Now from this equation we eliminate $y_{xx}$ by the first of equations (14), obtaining an equation involving $y_x$ with its second and third powers. Then from this new equation eliminate successively $(y_x)^3$ and $(y_x)^2$ by means of the second of equations (14), the work presenting no difficulty whatever, except its

length. By these operations, and retaining the second of equations (14), we shall have

$$\left.\begin{array}{r}\{V_r V_t' + V_s(V_q - V_s' - V_{t_,}) + V_t(V_{s_,} + 3V_r' - 2V_p)\}dx \\ + \{V_t' V_{r_,} + V_s(V_p - V_{s_,} - V_r') + V_r(V_s' + 3V_{t_,} - 2V_q)\}dy = 0, \\ V_r dy^2 - V_s dy\,dx + V_t dx^2 = 0.\end{array}\right\} \quad (19)$$

**425.** It is easy to show that conditions (14), or rather (19), must hold also along the *right* and *left* portions of the contour when they exist. For since along these portions $dx = 0$, the second of equations (19) will give $V_r = 0$; so that $V_{r_,} = 0$, and then these three conditions will cause the first of equations (19) to reduce to

$$V_p - V_r' - V_{s_,} = 0.$$

For the four corners of the required surface we merely join to equations (14) or (19) equation (16).

We might in the same manner as before discuss the case in which the required surface is to be limited by any given surface or surfaces. But as this examination would not prove useful, because of the scarcity of actual problems, and as it is believed that the reader will now be able to investigate these cases for himself, we shall proceed no further in the discussion of this subject.

**426.** We have now seen that the method of M. Sarrus enables us to investigate in a systematic manner the conditions which must, under any supposition, hold at the limits in order that $U$ may be a maximum or a minimum; and so far as this method itself is concerned, it should be regarded as satisfactory and sufficient. But while it gives the conditions which must prevail at the limits, if there be any solution, it still remains for us to determine whether or not these conditions can be fulfilled, and we shall find at this point that the

theory is much less satisfactory than in the case of simple integrals. For supposing $V$ to contain differential coefficients of $z$ to the order $n$ inclusive, we know that the equation $M = 0$ will be in general a partial differential equation of the order $2n$. Now we are rarely able to integrate an equation of this class, and are not certain that all such equations admit of any solution at all in finite terms; and even if we suppose a solution to exist, we cannot tell *a priori* how many arbitrary functions it must involve, all that we know being that the number of these functions will not exceed that which marks the order of the differential equation in question. Moreover, even if we knew the number of these functions, we could not say how many conditions they might be made to satisfy, since we would not know what should be the quantities under the functional sign. Also, when we have obtained an integral of one of these equations, we cannot be always certain that the solution is of the most general possible character.

**427.** From what has been said, it will appear that we cannot, as in the case of simple integrals, assert that because the equation $M = 0$ is of the order $2n$, the general solution can be subjected to $2n$ conditions at the limits; although the examination of particular cases, as well as the analogy of simple integrals, would lead us to infer such to be the case. If, for example, we require that the surface given by the equation $M = 0$ shall pass through $2n$ distinct curves, or shall have its edges upon $2n$ surfaces, we do not know that these conditions can be satisfied, but our inference that they can is supported by the following additional considerations.

In an equation of the form $M = 0$ we can assign arbitrarily the values of $z$ corresponding to $x = 0$ or to some function of $x$ and $y$ equals zero, and also those of the first $2n - 1$ differential coefficients of $z$ with respect to either variable, $x$ for example. Now by assigning the values of $z$ we compel the surface to pass through one given curve, which

would be all that we could do in the case of a partial differential equation of the first order. When the equation is of the second order, we can, as before, make the surface first pass through some curve, and then, by suitably assigning the value of $p$, can fix the position of the tangent plane along this curve; that is, can make the surface pass through two curves which are consecutive.

In like manner, when the equation is of the order $2n$, we can effect that the surface shall pass through $2n$ curves which are consecutive one to another; and since this can be done so long as the curves are indefinitely near one another, we may infer that it would also be possible if the curves were separated by finite spaces, although we must be careful not to speak with too much certainty upon this point.

The last two articles are due chiefly to Moigno and Lindelöf. See their *Calcul des Variations*.

**428.** The equation $M = 0$ will not, however, always rise to the order $2n$. If $V$ be a function involving $x, y, z, p$ and $q$ only, thus naturally making $M$ of the second order, it is readily shown that $M$ will not rise above the first order if $V$ have the form

$$V = f_1(x, y, z) + f_2(x, y, z)p + f_3(x, y, z)q, \tag{1}$$

and in this case only. But if $V$ contain $x, y, z, p, q, r, s$ and $t$, giving usually $M$ of the fourth order, it can be shown that to prevent $M$ from rising above the third order, it is necessary and sufficient that, $A$, $B$, $C$, $D$ and $E$ being severally functions of $x, y, z, p$ and $q$, $V$ shall be of the general form

$$V = A(rt - s^2) + Br + 2Cs + Dt + E. \tag{2}$$

Moreover, it is shown that in both these cases the equation $M = 0$ cannot in reality rise above the order $2n - 2$. See the work of Prof. Jellett, page 249.

It will be remembered that the corresponding case for

simple integrals arises from the fact that the integral $\int_{x_0}^{x_1} V dx$ is capable of some reduction by integration, and should be reduced before applying the calculus of variations. But we cannot extend the analogy. For in the present case no such reduction is, in general, possible.

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## SECTION IV.

### *EXTENSION OF JACOBI'S THEOREM TO THE DISCRIMINATION OF MAXIMA AND MINIMA OF DOUBLE INTEGRALS.*

**429.** We will now present a mere outline of the method of extending the theorem of Jacobi to double integrals, considering the case in which $V$ is a function of $x, y, z, p$ and $q$ only, and supposing, as usual, that the limiting values of $x, y$ and $z$ are fixed.

Now since the terms of the first order must vanish, if $U$ is to become a maximum or a minimum, we shall have

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\{V_{zz}\delta z^2 + 2V_{zp}\delta z\,\delta p + 2V_{zq}\delta z\,\delta q$$
$$+ 2V_{pq}\delta p\,\delta q + V_{pp}\delta p^2 + V_{qq}\delta q^2\}dy\,dx. \quad (1)$$

Now we can change the form of $\delta U$ thus:

$$\delta U =$$
$$\frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\{V_{zz}\delta z + V_{zp}\delta p + V_{zq}\delta q - (V_{zp}\delta z + V_{pp}\delta p + V_{pq}\delta q)'$$
$$- (V_{zq}\delta z + V_{pq}\delta p + V_{qq}\delta q)_{,}\}\delta z\,dy\,dx. \quad (2)$$

The truth of (2) can easily be verified by integrating once by parts each of the quantities within the accented pa-

rentheses, the first set with respect to $x$ and the second with respect to $y$, remembering that the limiting values of $z$ are fixed. Thus, for example,

$$-\int_{x_0}^{x_1}(V_{zp}\delta z)'\delta z\,dx = -\Big/_{x_0}^{x_1} V_{zp}\delta z^2 + \int_{x_0}^{x_1} V_{zp}\delta z\,\delta p\,dx.$$

Proceeding thus with each term, we shall obtain the same form for $\delta U$ as in (1).

Now let

$$B = V_{zz} - V_{zp}{}' - V_{zq_{,}}.$$

Then (2) may be written

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\{B\delta z - (V_{pp}\delta p + V_{pq}\delta q)'$$

$$- (V_{pq}\delta p + V_{qq}\delta q)_{,}\}\,\delta z\,dy\,dx. \qquad (3)$$

**430.** Thus it will appear, upon comparing equation (3) with equation (7), Art. 129, that $\delta U$ has been put under a form which we may call Jacobi's form for two independent variables. Moreover, it will appear, as in the case of simple integrals, that, because the limiting values of $x$, $y$ and $z$ are fixed, we must have

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\delta M\,\delta z\,dy\,dx; \qquad (4)$$

so that we have

$$\delta M = B\delta z - (V_{pp}\delta p + V_{pq}\delta q)' - (V_{pq}\delta p + V_{qq}\delta q)_{,}. \qquad (5)$$

Now let $u$ be such a quantity as will satisfy the equation

$$Bu - (V_{pp}u' + V_{pq}u_{,})' - (V_{pq}u' + V_{qq}u_{,})_{,} = 0. \qquad (6)$$

Then if $\delta z$ can be made equal to $u$ or $ku$ throughout the whole or a portion of the double integral, $k$ being an infinitesimal

constant, $\delta U$ to the second order can be made to vanish, and we would infer, as before, that $U$ is neither a maximum nor a minimum.

Now in (3) put $ut$ for $\delta z$. Then the resulting equation may be written

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1} Wt\,dy\,dx. \tag{7}$$

$$W = \left\{ B\,ut - \{V_{pp}(ut)' + V_{pq}(ut)_{,}\}' - \{V_{pq}(ut)' + V_{qq}(ut)_{,}\}_{,} \right\} u. \tag{8}$$

But because (6) is true, $W$ is integrable. For, multiplying (6) by $ut$ and subtracting from $W$ as in Art. 135, we have

$$-W = u\{V_{pp}(ut)'\}' - ut\{V_{pp}u'\}' + u\{V_{pq}(ut)_{,}\}' - ut\{V_{pq}u_{,}\}'$$
$$+ u\{V_{pq}(ut)'\}_{,} - ut\{V_{pq}u'\}_{,} + u\{V_{qq}(ut)_{,}\}_{,} - ut\{V_{qq}u_{,}\}_{,}. \tag{9}$$

Now proceeding with each pair of terms precisely as in Art. 135, it is evident that the first and last will give no trouble, and we shall also find that no difficulty will occur in the second, but $u'^2$ and $u_{,}^2$ will merely be replaced by $u_{,}'$. Proceeding then as indicated, we shall ultimately find

$$W = -\{(V_{pp}t' + V_{pq}t_{,})u^2\}' - \{(V_{pq}t' + V_{qq}t_{,})u^2\}_{,}. \tag{10}$$

Substituting this value in (7), and integrating by parts, one portion with respect to $x$, and the other with respect to $y$, observing that $t$ or $\frac{\delta z}{u}$ must vanish at the limits, we shall obtain

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\{(V_{pp}t' + V_{pq}t_{,})u^2t' + (V_{pq}t' + V_{qq}t_{,})u^2t_{,}\}dy\,dx$$

$$= \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\{V_{pp}t'^2 + 2V_{pq}t't_{,} + V_{qq}t_{,}^2\}u^2dy\,dx. \tag{11}$$

**431.** Now it will first of all be necessary to the existence of a maximum or a minimum that the coefficient of $u^2$ shall be of invariable sign throughout the field of the double integral. Putting $T$ for $\frac{t'}{t_i}$, this coefficient may be written

$$V_{pp}\left\{T^2+2\frac{V_{pq}}{V_{pp}}T+\frac{V_{qq}}{V_{pp}}\right\}T_i^2. \qquad (12)$$

But to secure that the middle factor of (12) shall be incapable of changing its sign or vanishing for any real value of $T$ positive or negative, the equation

$$T^2+2\frac{V_{pq}}{V_{pp}}T+\frac{V_{qq}}{V_{pp}}=0$$

must be incapable of being satisfied by any but two imaginary values of $T$; so that we must have

$$\frac{(V_{pq})^2}{(V_{pp})^2}<\frac{V_{qq}}{V_{pp}}.$$

Therefore it is necessary to the existence of a maximum or a minimum that $V_{pp}$ shall be of invariable sign throughout the portion of the double integral which we are considering, and also that $V_{pp}V_{qq}-(V_{pq})^2$ shall be always positive, although it may vanish at some point. By reference to works on the differential calculus it will appear that these conditions are analogous to those which must hold when we seek by the ordinary method to maximize or minimize a function of two variables which are independent.

**432.** But it is evident that before we can assert in any particular case that we have a maximum or a minimum, we must, after finding the two above conditions to be satisfactory, be able to show that $u$ or $ku$ is not an admissible value of $\delta z$ throughout any finite portion of the integral, and also that no

element of $\delta U$ will become infinite. To ascertain these points we must be able to determine the quantity $u$; and here the theory practically fails, although in theory $u$ may be determined in the following manner:

Suppose the equation $M = 0$ were completely integrable, the integrals being of such a form that we could obtain $z$ as a function of $x$ and $y$, and probably two arbitrary functions of $x$ and $y$, and then that by means of the conditions furnished by the fixed contour $z$ could be found as a known function of $x, y$ and two constants, say $z = f(x, y, c_1, c_2) = f$. If now we vary $c_1$ and $c_2$, the corresponding values of $\delta z$, $\delta p$ and $\delta q$, although not necessarily zero, will be such that $z + \delta z$, $p + \delta p$ and $q + \delta q$ will still satisfy the equation $M = 0$; that is, $\delta M$ will be zero. Therefore, because (5) is true, it will appear, by precisely the same reasoning as in Art 132, that

$$u = \frac{dV}{dc_1} + l\frac{dV}{dc_2}.$$

The preceding discussion is all that we have space to present, nor would a more extended treatment prove profitable. But the reader who may wish to pursue this subject further is referred to an article by A. Clebsch on the reduction of the second variation of a multiple integral, contained in the fifty-sixth volume of Crelle's *Mathematical Journal* for 1859.

## SECTION V.

### *MAXIMA AND MINIMA OF TRIPLE INTEGRALS.*

### Problem LXVI.

**433.** *Let $u$ be the density at any point of a body whose form, position and mass are known. Then, denoting by $p$, $q$ and $r$ the partial differentials $\frac{du}{dx}$, $\frac{du}{dy}$ and $\frac{du}{dz}$, it is required to determine the law of this density, so as to minimize the expression*

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\sqrt{1+p^2+q^2+r^2}\,dz\,dy\,dx. \tag{1}$$

Since the mass of the body is to remain constant, we must have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}u\,dz\,dy\,dx \tag{2}$$

always constant.

Now extending the method of double integrals, we always suppose that when $u$ is known as a function of $x$, $y$ and $z$, (1) and (2) are first integrated with reference to $z$ only, $x$ and $y$ being regarded as constants; and for this purpose we must first substitute in (2) the value of $u$ as a function of $x$, $y$ and $z$, and in (1) the values of $p$, $q$ and $r$ derived from this function. In other words, the body is supposed to be divided, by planes parallel to the co-ordinate planes, into prisms whose edges are $dz$, $dy$ and $dx$; and we first sum up these prisms along any ordinate $z$, or we, at any rate, obtain the portion of the integral comprised within this column.

Thus, considering (2), for example, we would have

$$dy\,dx\int_{z_0}^{z_1}u\,dz=\int_{z_0}^{z_1}f(x,y,z)\,dy\,dx, \tag{3}$$

which will give us the solidity of any right parallelepipedon whose edges are $z_1 - z_0$, $dy$ and $dx$.

Before we can proceed further with the integration, $z_0$ and $z_1$ must be determined as functions of $x$ and $y$; that is, the body must be limited in the direction of the $z$'s by surfaces whose equations are $z = z_0$ and $z = z_1$, $z_0$ and $z_1$ being known functions of $x$ and $y$. After this substitution, (3) may be written

$$dy\,dx\int_{z_0}^{z_1} u\,dz = f'(x,y)\,dy\,dx. \tag{4}$$

We next integrate with reference to $y$ only; that is, we sum up all the parallelepipedons corresponding to any particular value of $x$, and thus obtain

$$dx\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dy = \int_{y_0}^{y_1} F(x,y)\,dx, \tag{5}$$

which will give us the solidity of a section of the thickness $dx$, and cut from the body by two planes at right angles to $x$.

Now the values $y_0$ and $y_1$ corresponding to any particular value of $x$, are the limits of this section in the direction of the $y$'s. But instead of supposing any section to terminate in a point, we shall suppose it to be terminated by a right line perpendicular to the plane of $xy$, because this supposition is more general, the former being at once deducible from it by merely reducing these terminal lines to points.

Hence, under the most general supposition, the body is supposed to be limited in the direction of the $y$'s by certain cylindrical walls whose equations are $y = y_0$ and $y = y_1$; and therefore before integrating again we must determine $y_0$ and $y_1$ as functions of $x$, and substitute this value in (5). Then (5) may be written

$$dx\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dy = F'(x)\,dx. \tag{6}$$

We now integrate with respect to $x$; that is, sum up the sections just mentioned. This will give

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dy\,dx = \int_{x_0}^{x_1} F'dx = \Big/_{x_0}^{x_1} F''(x). \qquad (7)$$

Now the most general supposition is that the body is terminated in the direction of the $x$'s by two planes perpendicular to $x$, whose equations are $x = x_0$ and $x = x_1$. For otherwise it can only terminate in an edge perpendicular to $x$, or in a point, both of which cases are at once deducible from the first supposition.

**434.** Thus as a geometrical conception we may consider any definite triple integral as extending throughout the entire space comprised within six faces; the first two, $z = z_0$ and $z = z_1$, which we shall denote by $C_0$ and $C_1$, being of any character whatever, either face being, if necessary, made up of surfaces satisfying different equations; the second two being the cylindrical walls $y = y_0$ and $y = y_1$, which we shall denote by $B_0$ and $B_1$, either face being at liberty to become merely a continuous or discontinuous edge, or to be composed of different cylindrical faces whose generatrices are parallel to the axis of $z$; the third two, which we shall denote by $A_0$ and $A_1$, being merely the planes $x = x_0$ and $x = x_1$, where either plane may reduce to a point or to any right line perpendicular to $x$.

**435.** Now suppose that throughout the solid given by (2) we make, according to some law, an infinitesimal change in the density $u$. Then we shall obtain a new solid which, while not differing from the first in form, will differ in its molecular condition, and may be called the derived solid. Moreover, to obtain the difference in the masses of these solids, we have merely to sum up the changes which take place in each element $u\,dz\,dy\,dx$.

But in varying any element, it is most natural to consider

the parallelepipedon as undergoing no change whatever in position or form, but in density only. That is, we may regard $x, y$ and $z$, and consequently $dx$, $dy$ and $dz$, the edges of the parallelepipedon, as incapable of variation. Hence, the mass being $m$, we would have

$$\delta m = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \delta u \, dz\, dy\, dx.$$

Now, to generalize, let

$$U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V\, dz\, dy\, dx,$$

where $V$ is any function of $x, y, z, u, p, q$ and $r$. Then it will appear, by the same reasoning as before, that when we vary $u$, $p$, $q$ and $r$, the limiting values of $x, y$ and $z$—that is, the limiting faces—being fixed, the corresponding change in $U$ will be

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \{V_u \delta u + V_p \delta p + V_q \delta q + V_r \delta r\}\, dz\, dy\, dx. \quad (8)$$

Moreover, it is evident that if $V$ contain differential coefficients to any order, the same method must be pursued in obtaining $\delta U$.

**436.** Now let us first assume $u$ to undergo no variation at the limits; that is, along the six faces. Or, to fix our ideas, suppose that in equation (1) or (2) the density were required to remain fixed throughout all the limiting faces, or the surface of the body, it being assigned by us arbitrarily for each limiting surface at the outset. Then $\delta u$ will vanish at the limits, and it is evident that by integrating by parts we can obtain, from (8),

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \left\{ V_u - \frac{dV_p}{dx} - \frac{dV_q}{dy} - \frac{dV_r}{dz} \right\} \delta u\, dz\, dy\, dx$$

$$= \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} M \delta u\, dz\, dy\, dx. \quad (9)$$

In like manner we may treat the case in which $V$ contains differential coefficients of $u$ to any order.

**437.** Returning to (8), let us now consider how to transform the variation of $U$ when $u$ is unrestricted at the limits.

By equation (7), Art. 382, we have, putting

$$\delta u = t, \qquad \delta p = \frac{dt}{dx}, \qquad V_p = u,$$

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V_p \delta p \, dz\, dy\, dx =$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \frac{dV_p}{dx}\delta u \, dz\, dy\, dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V_p \,\delta u\, dz\, dy$$

$$-\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1} V_p \frac{dy}{dx}\delta u\, dz\, dx - \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} V_p \frac{dz}{dx}\delta u\, dy\, dx. \quad (10)$$

Again, writing $V_q = u$, $\delta q = \dfrac{dt}{dy}$, we have, by (8), Art. 383,

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V_q \,\delta q\, dz\, dy\, dx = -\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \frac{dV_q}{dy}\delta u\, dz\, dy\, dx$$

$$+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1} V_q\, \delta u\, dz\, dx - \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} V_q \frac{dz}{dy}\delta u\, dy\, dx. \quad (11)$$

Lastly, by equation (9), Art. 384, or by integrating directly by parts, we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V_r \delta r\, dz\, dy\, dx =$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \frac{dV_r}{dz}\delta u\, dz\, dy\, dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} V_r \delta u\, dy\, dx. \quad (12)$$

Substituting these values in (8), and arranging, we have

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1}\left\{V_r - V_p\frac{dz}{dx} - V_q\frac{dz}{dy}\right\}\delta u\, dy\, dx$$

$$+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{V_q - V_p\frac{dy}{dx}\right\}\delta u\, dz\, dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V_p\,\delta u\, dz\, dy$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{V_u - \frac{dV_p}{dx} - \frac{dV_q}{dy} - \frac{dV_r}{dz}\right\}\delta u\, dz\, dy\, dx. \qquad (13)$$

In this form of $\delta U$ we observe that there are four terms, the first relating to the faces $C_0$ and $C_1$ only; the second to the cylinders $B_0$ and $B_1$ only; the third to the planes $A_0$ and $A_1$ only; while the fourth extends throughout the entire integral $U$.

**438.** Now when $U$ is to be a maximum or a minimum, $\delta U$ to the first order must vanish; and it is evident that whether the terms at the limits exist or not, we shall obtain the equation

$$M = V_u - \frac{dV_p}{dx} - \frac{dV_q}{dy} - \frac{dV_r}{dz} = 0, \qquad (14)$$

an equation which holds throughout the integral. The integral of this equation will involve certain arbitrary functions; but if $u$ be invariable at the limits, the calculus of variations affords us no further equations of condition, and these functions must be determined by the values which we require $u$ to maintain upon the various faces which limit the integral.

But when $u$ is unrestricted at the limits, we must also have $L = 0$, $L$ denoting the terms at the limits; and since $\delta u$ (for example, the variation of the density, if, as in (1) and (2), $u$ be the density) is entirely in our power for each point of the

various faces, these faces themselves undergoing no change in form, it is easy enough to see that we must have the equations

$$\left.\begin{array}{l} V_r - V_p \dfrac{dz}{dx} - V_q \dfrac{dz}{dy} = 0, \\ \\ V_q - V_p \dfrac{dy}{dx} = 0, \quad V_p = 0, \end{array}\right\} \qquad (15)$$

the first equation holding for the faces $C_0$ and $C_1$, the differentials $\frac{dz}{dx}$ and $\frac{dz}{dy}$ being relative to these faces only; the second holding for the walls $B_0$ and $B_1$, the differential $\frac{dy}{dx}$ being relatives to these walls only; and the third holding only throughout the planes $A_0$ and $A_1$.

**439.** We will now show that, as in the case of double integrals, these conditions are in reality identical.

Let $a$, $b$ and $c$ denote the angles made with the co-ordinate axes by any normal to $C_0$ or $C_1$. Then the first of equations (15) will give the conditions

$$V_p \cos a + V_q \cos b + V_r \cos c = 0. \qquad (16)$$

Now for the cylindrical faces $B_0$ and $B_1$ we have $\cos c = 0$, and the second of equations (15) would therefore give the condition

$$V_p \cos a + V_q \cos b = 0, \qquad (17)$$

which would follow at once from (16) when $\cos c$, vanishes. Lastly, for the planes $A_0$ and $A_1$, we have $\cos c = 0$, $\cos b = 0$ and $\cos a = 1$; so that equation (16) would give at once $V_p = 0$, the equation required. It appears, therefore, that the first of equations (15), or rather that equation (16), holds for all the faces.

**440.** Now it is clear that the problem proposed at the beginning of this section is one of relative minima, since the variations of $u$, $p$, $q$ and $r$ are to be so restricted as to permit a certain mass to remain constant. But it will readily appear that by selecting portions of the solid, just as we did of the curve for single integrals and of the surface for double integrals, we can extend the method of Euler, by the reasoning of Bertrand, to triple integrals also; and we shall therefore assume this fact without further discussion.

Now assuming, for convenience, $-\frac{1}{g}$ as the constant multiplier, we must, as we see from (1) and (2), minimize the expression

$$U=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{\sqrt{1+p^2+q^2+r^2}-\frac{u}{g}\right\}dz\,dy\,dx$$

$$=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}V dz\,dy\,dx. \tag{18}$$

Here

$$\left.\begin{aligned} V_u&=-\frac{1}{g}, & V_p&=\frac{p}{\sqrt{1+p^2+q^2+r^2}},\\ V_q&=\frac{q}{\sqrt{1+p^2+q^2+r^2}}, & V_r&=\frac{r}{\sqrt{1+p^2+q^2+r^2}};\end{aligned}\right\} \tag{19}$$

so that the equation $M=0$ gives

$$\frac{1}{g}+\frac{d}{dx}\frac{p}{\sqrt{1+p^2+q^2+r^2}}+\frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2+r^2}}$$

$$+\frac{d}{dz}\frac{r}{\sqrt{1+p^2+q^2+r^2}}=0. \tag{20}$$

According to Moigno, one solution of this differential equation is

$$(x-h)^2+(y-i)^2+(z-j)^2+(u-k)^2=9g^2, \tag{21}$$

$h$, $i$, $j$ and $k$ being constants.

This is an equation which is analogous to that of the circle and sphere. If we suppose $u$ fixed at the limits, we must assign it so as not to conflict with this equation. These conditions could evidently all be fulfilled if the body were a sphere the density of whose surface were to remain uniform throughout.

When $u$ is not restricted at the surface of the body, equation (16) will give at once the condition

$$p \cos a + q \cos b + r \cos c = 0, \tag{22}$$

which, as we have seen, must hold for all the limiting faces.

**441.** Let us now suppose that while the mass of the body is to remain constant, its form is not fixed, its density at the surface, however, being required always to satisfy the equation

$$f(x, y, z, u) = f = 0, \tag{23}$$

$f$ being any function whatever.

This case corresponds, for the triple integral, to that of a double integral in which the required surface is to have its contour always upon one or more given surfaces. For here the faces $C_0$, etc., take the place of the contour, $u$ is the dependent variable instead of $z$, and the limiting function or functions $f = 0$ which $u$ is to satisfy upon the faces, take the place of the equation or equations of the surface or surfaces upon which the contour must rest.

But to enable us to discuss this case, we must first consider how to find the variation of $U$ when the limiting values of $x$, $y$ and $z$, as well as those of $u$, are variable, where

$$U = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} V dz\, dy\, dx,$$

$V$ being any function of $x$, $y$, $z$, $u$, $p$, $q$ and $r$. We can evidently pass from the primitive to any derived solid by first

varying the limiting faces, supposing $u$ to remain unvaried, and then varying $u$ throughout the new solid. In varying the faces, suppose first that $x$ and $y$ remain constant, and that we change any two ordinates $z_0$ and $z_1$ of the faces $C_0$ and $C_1$ into $z_0 + Dz_0$ and $z_1 + Dz_1$. Then, by precisely the same reasoning as if $z$ were the only independent variable, as indeed it is so long as we are passing from $z_0$ to $z_1$ only, we shall have, to the first order, as the change in $\int_{z_0}^{z_1} Vdz$, $V_1Dz_1 - V_0Dz_0$; and extending this method to every value $z_0$ and $z_1$, we shall have, as the part of $\delta U$ resulting from the variation of the faces $C_0$ and $C_1$,

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V\,Dz\,dy\,dx. \tag{24}$$

We may next vary the walls $B_0$ and $B_1$, supposing $x$, $z$ and $u$ to remain unvaried. But in varying these faces we must remember that they are always to remain cylindrical, every generatrix being parallel to the axis of $z$. Now, as before, when we change $y_0$ into $y_0 + Dy_0$, and $y_1$ into $y_1 + Dy_1$, $x$ and $z$ remaining unvaried, $y$ is the only independent variable, and the corresponding change in $\int_{y_0}^{y_1} Vdy$ will, to the first order, be $V_1Dy_1 - V_0Dy_0$; and this, being taken throughout any generatrix, will give $\int^{y_0}\int_{z_0}^{z_1} V\,Dy\,dz$, where $Dy$ is a constant throughout the generatrix in question, but independent for each. Now since we can only vary $B_0$ and $B_1$ by varying each generatrix in the manner described, we must take the sum of these variations, the integration being with respect to $x$; and therefore the change resulting to $U$ from varying $B_0$ and $B_1$ will be

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V\,Dy\,dz\,dx. \tag{25}$$

We next vary the planes $A_0$ and $A_1$, supposing $y$, $z$ and $u$ to remain unvaried, and keeping $A_0$ and $A_1$ always planes, per-

pendicular to the axis of $x$. Then the change of $x_0$ and $x_1$ into $x_0 + Dx_0$ and $x_1 + Dx_1$ will give $V_1Dx_1 - V_0Dx_0$. Hence, extending this method to every point of the planes $A_0$ and $A_1$, we shall obtain

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V\,Dx\,dz\,dy, \tag{26}$$

where $Dx_0$ and $Dx_1$ are constants, but independent.

Besides those which we have evidently omitted, various other terms of the second and higher orders would arise at the intersection of the faces, which we do not propose to consider.

**442.** If now, in the second place, having varied the faces, we vary $u$ throughout the new limits, these limiting faces themselves remaining fixed, it is evident that the result cannot differ by any term of the first order from the value of $\delta U$ before the limits were varied, the difference consisting only of the variations of (24), (25) and (26), themselves quantities of the first order only. Hence this part of $\delta U$ is to be found and transformed as already explained. Therefore, finally, adding (13), (24), (25) and (26), we shall have

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \left\{ V_r - V_p\frac{dz}{dx} - V_q\frac{dz}{dy} \right\} \delta u\,dy\,dx$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \left\{ V_q - V_p\frac{dy}{dx} \right\} \delta u\,dz\,dx$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V_p\,\delta u\,dz\,dy + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V\,Dz\,dy\,dx$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V\,Dy\,dz\,dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V\,Dx\,dz\,dy$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \left\{ V_u - \frac{dV_p}{dx} - \frac{dV_q}{dy} - \frac{dV_r}{dz} \right\} \delta u\,dz\,dy\,dx = 0. \tag{27}$$

Moreover, it is evident that if $V$ contain differential coefficients of $u$ to any order, we shall still obtain the most general value of $\delta U$ by merely adding (24), (25) and (26) to the result obtained by varying $U$, supposing the limiting values of $x$, $y$ and $z$ to be fixed.

**443.** Now if $U$ is to become a maximum or a minimum, we must, denoting the terms at the limits by $L$, equate $L$ and $M$ severally to zero, where $M$ is the same as before, but $L$ is not. Then, if we could suppose the quantities $Dz$, $Dy$ and $Dx$ to be entirely independent of $\delta u$, we shall first of all obtain from $L = 0$ equations (15), besides which, if $Dz$, $Dy$ and $Dx$ be independent, we must equate severally to zero (24), (25) and (26). But since $Dz_0$ and $Dz_1$ have all the independence of variations, (24) will give $V = 0$, which, together with the first of equations (15), must hold throughout the curved surfaces $C_0$ and $C_1$.

Now in (25) $Dy$ is an independent constant along each several generatrix, so that each element of the integral must vanish, and we must have $\int_{z_0}^{z_1} VDy\,dz = 0$, which holds along any single generatrix only; and $Dy$ being constant along this generatrix, we have $\int_{z_0}^{z_1} Vdz = 0$, which is all the reduction we can effect, and must hold along each generatrix of the faces $B_0$ and $B_1$, while the second of equations (15) holds for each point of these faces.

Lastly, in (26) $Dx_0$ and $Dx_1$ are two independent constants, and we have, therefore, $\int_{y_0}^{y_1}\int_{z_0}^{z_1} Vdz\,dy = 0$, which is the final equation, and holds throughout the entire planes $A_0$ and $A_1$ only, while the third of equations (15) must hold for each separate point of these faces.

**444.** Now although we do not know, *a priori*, just how many conditions at the limits the solution of the equation

$M = 0$ can be made to satisfy, the number will probably not exceed two, so that it is evident that the quantities $\delta u$, $Dz$, $Dy$ and $Dx$ cannot be regarded as entirely independent, since this supposition would, as we have seen, give us twelve equations at the limits, two relative to each of the six faces. Hence we must impose some restriction at the limits of the kind mentioned in the beginning of Art. 441, which we now proceed to consider, but at first in a general manner, and not relative to any particular problem.

**445.** Suppose, first, that upon the face $C_1$, $u$ is required to equal some function $f$ of $x$, $y$ and $z$, this face itself being subject to variation of form; and let $P$, $Q$ and $R$ be the partial differential coefficients of $f$ with respect to $x$, $y$ and $z$. Then, because, when we pass along any ordinate $z$, $x$ and $y$ remain constant, $f$ becomes in reality a function of $z$ and constants only, and might, therefore, be made the ordinate of a plane curve, $z$ being the abscissa. Hence, by the same reasoning as hitherto, we must have

$$/^{z_1}\delta u = /^{z_1}(R - r)\,Dz, \tag{28}$$

and a similar equation for the lower limit of $z$. In like manner. if we suppose that $u$ must equal $f$ upon each of the other faces, it being immaterial whether or not $f$ be the same function for all the faces, or even throughout the same face, we shall, by extending to $x$ and $y$ the reasoning just employed for $z$, obtain for $B_1$ and $A_1$ respectively,

$$/^{y_1}\delta u = /^{y_1}(Q - q)\,Dy, \qquad /^{x_1}\delta u = /^{x_1}(P - p)\,Dx, \tag{29}$$

with similar equations for the lower limits.

Now taking the value of $L$ from (27), putting together the terms affected by like signs of substitution, and then eliminating $\delta u$ from each by equations (28) and (29), we shall have

$$L = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{y_0}^{z}\left\{V+\left(V_r - V_p\frac{dz}{dx} - V_q\frac{dz}{dy}\right)(R-r)\right\}Dz\,dy\,dx$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{V+\left(V_q - V_p\frac{dy}{dx}\right)(Q-q)\right\}Dy\,dz\,dx$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\{V+V_p(P-p)\}Dx\,dz\,dy = 0. \tag{30}$$

**446.** Now, because the quantities $Dz_0$, $Dz_1$, $Dy_0$, $Dy_1$, $Dx_0$ and $Dx_1$ are all independent, we must, in the first place, equate severally to zero each of the three, or rather six terms in $L$. Hence, as before, the first will evidently give

$$V+\left(V_r - V_p\frac{dz}{dx} - V_q\frac{dz}{dy}\right)(R-r) = 0, \tag{31}$$

an equation which relates to either of the faces $C_0$ or $C_1$ only.

Next, by the same reasoning as before, the second term can only be made to give

$$\int_{z_0}^{z_1}\left\{V+\left(V_q - V_p\frac{dy}{dx}\right)(Q-q)\right\}dz = 0, \tag{32}$$

an equation which holds for any one generatrix only of the faces $B_0$ or $B_1$, the integral being required to be taken throughout that entire generatrix.

Lastly, the third term will give

$$\int_{y_0}^{y_1}\int_{z_0}^{z_1}\{V+V_p(P-p)\}dz\,dy = 0, \tag{33}$$

an equation which holds for the planes $A_0$ or $A_1$ only, the integration being required to extend throughout the entire surface of either plane.

But since $u$ must always equal $f$ upon the face $C_0$ or $C_1$, if

we pass from one point to another upon either of these faces, the change which $u$, the density, for example, will undergo, will equal the corresponding change in $f$, however these points may be situated. Let us, then, assume these points to be indefinitely near each other, and both to be first in a section cut by a plane parallel to that of $xz$, and then in one cut by a plane parallel to that of $yz$. These suppositions will give respectively

$$p\,dx + r\,dz = P\,dx + R\,dz, \qquad q\,dy + r\,dz = Q\,dy + R\,dz.$$

Whence

$$\frac{dz}{dx} = -\frac{P-p}{R-r}, \qquad \frac{dz}{dy} = -\frac{Q-q}{R-r}. \tag{34}$$

By similar reasoning, since $u$ always equals $f$ upon either cylindrical wall $B_0$ or $B_1$, if we pass from one of two consecutive points to the other along the intersection of this wall with the plane of $xy$, we must have

$$p\,dx + q\,dy = P\,dx + Q\,dy, \qquad \frac{dy}{dx} = -\frac{P-p}{Q-q}; \tag{35}$$

which, being true along the plane of $xy$, is of course true for the entire face $B_0$ or $B_1$.

Now substituting these values in (31) and (32), and reproducing (33), we have

$$\left.\begin{aligned} V + V_p(P-p) + V_q(Q-q) + V_r(R-r) &= 0, \\ \int_{z_0}^{z_1}\{V + V_p(P-p) + V_q(Q-q)\}\,dz &= 0, \\ \int_{y_0}^{y_1}\int_{z_0}^{z_1}\{V + V_p(P-p)\}\,dz\,dy &= 0. \end{aligned}\right\} \tag{36}$$

**447.** Such, then, are the equations which must hold for the various limiting faces; and the reader can easily apply

them to the particular problem with which we started, although no results of importance present themselves. Indeed, we assumed this problem merely because it better fixes our ideas to think at first of $u$, the dependent variable, as something physical or geometrical, like density, than as some function merely of $x$, $y$ and $z$, although the latter view will, in general, be necessary.

It would appear that, without reducing the number of the limiting faces, we shall still have in general too many equations at the limits, although our imperfect knowledge as to what should be the form of the most general possible solution of the equation $M = 0$ will prevent us from determining how far these conditions might be fulfilled.

Although the converse need not be true, the second and third of equations (36) would be satisfied should the first hold throughout all the limiting faces. For since $u = f$ along any particular generatrix of the $B$'s, $R$ and $r$ must be always equal along that generatrix; which would satisfy the second equation by giving

$$V + V_p(P - p) + V_q(Q - q) = 0.$$

In like manner, because $u = f$ throughout the planes $A_0$ and $A_1$, we see, by first passing along any line perpendicular to the plane of $xy$, and then along any line perpendicular to that of $xz$, that throughout the $A$'s, $R = r$, $Q = q$, and the first equation satisfies the third by giving

$$V + V_p(P - p) = 0.$$

**448.** When the limiting values of $x$, $y$, $z$ and $u$ are all fixed, the terms of the second order can be sometimes examined. Thus in the particular problem with which we opened this section, $V_{uu}$, $V_{up}$, $V_{uq}$ and $V_{ur}$ all reduce to zero,

the ten terms of the second order will reduce to six, and we shall easily find that $\delta U$ may be exhibited thus:

$$\delta U = \frac{1}{2}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\frac{1}{(1+p^2+q^2+r^2)^{\frac{3}{2}}}\{\delta p^2+\delta q^2+\delta r^2$$

$$+(q\delta p-p\delta q)^2+(r\delta p-p\delta r)^2+(r\delta q-q\delta r)^2\}dz\,dy\,dx,$$

which is evidently positive, thus giving us a minimum.

The foregoing discussion will render the reader sufficiently familiar with the treatment of triple integrals, while a discussion of those of a higher order would be of no use, except as a matter of curiosity, and would be beyond the design and scope of this work.

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## Section VI.

### *ANOTHER VIEW OF VARIATIONS.*

**449.** If, in the preceding discussion, we had for double integrals ascribed variations to $x$ and $y$, and for triple to $x$, $y$ and $z$, we could, as in the case for single integrals, have obtained the same formulæ as by the method which we have adopted. Or we might even, as Prof. Jellett does, in the case of double integrals, assume the required surface as the independent variable, considering $x$, $y$ and $z$ as functions of the surface.

But there is besides these another more analytical view of variations, applicable to integrals of any order, which presented itself to Euler and Lagrange, has been followed essentially not only by Strauch, but by Sarrus, and subsequently by Moigno and Lindelöf, as will appear from their *Calcul des Variations*, Leçon III.

**450.** To begin, then, with the simplest case, suppose a plane curve whose equation is $y = f(x) = f$, and change $y$ into

$y+\delta y$ or $Y$, $x$ remaining unvaried. Then we may regard $Y$ as a function $F$ of $x$ and $t$, where $t$ is a new quantity entirely independent of $x$, and constant, and may enter $F$ in any manner we please, provided only that the form of $F$ shall be such as to cause it to reduce to $f$ when $t$ is made zero. Then it is evident that if we regard $f$ as containing $t$, we must regard it as a function of $x$ and 0. Then, since $x$ does not vary for any change in $t$, we may, by Maclaurin's theorem, develop $Y$ in ascending powers of $t$, obtaining

$$Y=f+t\left[\frac{dF}{dt}\right]+\frac{t^2}{2}\left[\frac{d^2F}{dt^2}\right]+\text{etc.},\tag{1}$$

where brackets denote that $t$ is made zero after the differential coefficients of $F$ with respect to $t$ have been found. If we suppose $t$ to be made infinitesimal, we may neglect powers of $t$ of an order higher than the first, and write

$$Y-y=\delta y=\left[\frac{dF}{dt}\right]t,\tag{2}$$

in which, because $t$ may enter $F$ in any manner we please which will cause $F$ to reduce to $f$ when $t$ is zero, $\left[\frac{dF}{dt}\right]$ is entirely in our power, and may be made to become any function we please. Therefore, replacing $F$ by $y$, we have

$$\delta y=\left[\frac{dy}{dt}\right]t,\tag{3}$$

where $\delta y$ is as unrestricted as formerly.

In like manner, since, when $y=f$, $y'$ or $\frac{dy}{dx}=f'(x)=f'$, and when $y$ is supposed to be $F(x, t)$, $y'$ becomes $F'(x, t)$, we shall find

$$\delta y'=\left[\frac{dy'}{dt}\right]t.\tag{4}$$

But $t$ being independent of $x$, if we differentiate (3) with respect to $x$, we have

$$\frac{d\delta y}{dx} = t\frac{d}{dx}\left[\frac{dy}{dt}\right] = t\left[\frac{d}{dx}\frac{dy}{dt}\right] = t\left[\frac{dy'}{dt}\right].$$

Hence, as before,

$$\delta y' = \frac{d\delta y}{dx}, \tag{5}$$

and similarly we find $\delta y''$, $\delta y'''$, etc., to be as usual.

**451.** Next let $V$ be any function of $x, y, y', y''$, etc. Then when $y, y', y''$, etc., which are all functions of $x$, are supposed to become such arbitrary functions of $x$ and $t$ as will reduce them to their original values when $t$ is made zero, $V$ must also become some function of $x$ and $t$, and we have at once, as in the case of $y$ and $y'$,

$$\delta V = \left[\frac{\delta V}{dt}\right]t = \left\{\frac{dV}{dy}\left[\frac{dy}{dt}\right] + \frac{dV}{dy'}\left[\frac{dy'}{dt}\right] + \text{etc.,}\right\}t$$

$$= \frac{dV}{dy}\delta y + \frac{dV}{dy'}\delta y' + \text{etc.,} \tag{6}$$

which, to the first order, is the usual form.

Now let $V$ besides $y$ contain another dependent variable $z$ with its differential coefficients $z'$, $z''$, etc., with respect to $x$. Then if $y$ and $z$ are independent, $t$ may enter either in any arbitrary manner which will permit it, and also its differential coefficients, to remain unchanged when $t$ is zero. But if $y$ and $z$ are always to be connected by an equation, differential or other—that is, if $\delta y$ and $\delta z$ are to be related—$t$ may enter one of these quantities in any arbitrary manner, but must enter the other in such a way as to cause $y$ and $z$ to be related in the required manner. In either case, since $y$ and $z$ are func-

tions of $x$, when they become functions of $x$ and $t$, $V$ becomes also a function of $x$ and $t$; and proceeding as before, we have

$$\delta V = \left[\frac{dV}{dt}\right] t = \frac{dV}{dy}\left[\frac{dy}{dt}\right] t + \frac{dV}{dy'}\left[\frac{dy'}{dt}\right] t + \text{etc.}$$

$$+ \frac{dV}{dz}\left[\frac{dz}{dt}\right] t + \frac{dV}{dz'}\left[\frac{dz'}{dt}\right] t + \text{etc.}$$

$$= \frac{dV}{dy}\delta y + \frac{dV}{dy'}\delta y' + \text{etc.} + \frac{dV}{dz}\delta z + \frac{dV}{dz'}\delta z' + \text{etc.}, \quad (7)$$

which is the same form as before, and in it, as we have already seen, the variations may be independent or may be made to fulfil any conditions which we may impose.

In like manner, if $V$ be a function of any number of independent variables $x, y, z$, etc., and a dependent variable $u$ together with its differential coefficients with respect to $x, y, z$, etc., we shall, by supposing $u$ to become a function of $x, y, z$, etc., and $t$, instead of $x, y, z$, etc., only,—$t$ being entirely independent of these variables,—render $V$ a function of $x, y, z$, etc., and $t$, and obtain

$$\delta V = \left[\frac{dV}{dt}\right] t, \quad (8)$$

where the second member will always be the same as we would, by the ordinary method, obtain as the variation of $V$ when $x$, $y$ and $z$ do not vary.

**452.** Now suppose we have the equation $U = \int_{x_0}^{x_1} V dx$, $V$ being any function of $x, y, y', y''$, etc. Then $U$ will be some function of $x_0$ and $x_1$; and when $y$ becomes such a function of $x$ and $t$ as will reduce $Y$ to $y$ when $t$ is zero, $U$ will also become such a function of $x_0$, $x_1$ and $t$ as will reduce $U + \delta U$ to

$U$ by making $t$ zero. Therefore, proceeding as formerly, we must, to the first order, have

$$\delta U = \left[\frac{dU}{dt}\right] t, \tag{9}$$

where $\frac{dU}{dt}$ denotes the differentiation of $U$ with respect to $t$, and to everything which in any way depends upon $t$, and nothing more.

Now, in the most general case, we cannot regard the limiting values of $x$ as fixed, but we may suppose these also to become functions of $t$, together with some constants independent of $t$, and this supposition will give us all needed generality. In (9) the increments $\delta x_0$ and $\delta x_1$ or $Dx_0$ and $Dx_1$ become

$$/^{v_0}\left[\frac{dx}{dt}\right] t \quad \text{and} \quad /^{v_1}\left[\frac{dx}{dt}\right] t.$$

Now by equation (1), Art. 375, (9) gives

$$\delta U = t\left[\frac{d}{dt}\int_{x_0}^{x_1} Vdx\right] = \int_{x_0}^{x_1}\left[\frac{dV}{dt}\right] t\,dx + /_{x_0}^{x_1} V\left[\frac{dx}{dt}\right] t$$

$$= \int_{x_0}^{x_1}\delta V dx + /_{x_0}^{x_1} VDx,$$

where $\delta V$ has the form given in (6), and $\delta y'$, $\delta y''$, etc., are, as appears from (5), capable of the same transformation by integration as usual.

Moreover, if $V$ contain the dependent variable $z$ also, and its differential coefficients, the last equation will still hold, only $\delta V$ will take the form given in (7), and may be transformed as in the ordinary case of two dependent variables; so that for single integrals we always obtain at once the same

fundamental and the same limiting equations as by the ordinary method.

**453.** The reader will probably now be ready to admit that equation (9) must hold also for any definite multiple integral whose limits are fixed or variable.

Now in equation (1), Art. 375, change $u$ into $\int_{y_0}^{y_1} u\,dy$. Then we have, by the aid of the same equation,

$$\frac{d}{dt}\int_{x_0}^{x_1}\int_{y_0}^{y_1} u\,dy\,dx =$$

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{du}{dt}\,dy\,dx + \int_{x_0}^{x_1}\Big/_{y_0}^{y_1} u\frac{dy}{dt}\,dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} u\frac{dx}{dt}\,dy; \quad (10)$$

and here changing $u$ into $\int_{z_0}^{z_1} u\,dz$, we obtain

$$\frac{d}{dt}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dy\,dx =$$

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\frac{du}{dt}\,dz\,dy\,dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} u\frac{dz}{dt}\,dy\,dx$$

$$+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1} u\frac{dy}{dt}\,dz\,dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\frac{dx}{dt}\,dz\,dy. \quad (11)$$

Hence it is easy to see that we would have respectively for

$$u = \int_{x_0}^{x_1}\int_{y_0}^{y_1} V\,dy\,dx \quad \text{and} \quad u = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V\,dz\,dy\,dx$$

the two equations

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\delta V\,dy\,dx + \int_{x_0}^{x_1}\Big/_{y_0}^{y_1} V\,Dy\,dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1} V\,Dx\,dy,$$

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\delta V\,dz\,dy\,dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} V\,Dz\,dy\,dx$$

$$+ \int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1} V\,Dy\,dz\,dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V\,Dx\,dz\,dy,$$

which are the forms previously obtained for variable limits, and $\left[\frac{dV}{dt}\right]t$ or $\delta V$ is just what it would be by the ordinary method, and is transformable in the same manner; so that here also we shall obtain at once the same general and the same limiting equations by either method.

**454.** Before proceeding we shall require some additional formulæ in the calculus of substitution.

In (10) change $u$ into $\Big/_{z_0}^{z_1} u$. Then by equation (2), Art. 376, we obtain

$$\frac{d}{dt}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} u\,dy\,dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1}\left\{\frac{du}{dt} + \frac{du}{dz}\frac{dz}{dt}\right\}dy\,dx$$

$$+ \int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\Big/_{z_0}^{z_1} u\frac{dy}{dt}\,dx + \Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} u\frac{dx}{dt}\,dy. \qquad \text{(A)}$$

In equation (2), Art. 376, change $u$ into $\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dy$. Then finding the values of

$$\frac{d}{dt}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dy \quad \text{and} \quad \frac{d}{dx}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dy,$$

which may be done from (10) by changing $x$ to $y$ and $y$ to $z$, we have

$$\frac{d}{dt}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dy = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{\frac{du}{dt}+\frac{du}{dx}\frac{dx}{dt}\right\}dz\,dy$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\left\{\frac{dz}{dt}+\frac{dz}{dx}\frac{dx}{dt}\right\}dy$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\left\{\frac{dy}{dt}+\frac{dy}{dx}\frac{dx}{dt}\right\}dz. \qquad \text{(B)}$$

In equation (1), Art. 375, first change $u$ into $\int_{y_0}^{y_1} u$, and employ equation (2), Art. 376, with $y$ put for $x$. Then in the resulting equation change $u$ into $\int_{z_0}^{z_1} u\,dz$, and reduce by equation (1), Art. 375, with $z$ put for $x$. Then we obtain

$$\frac{d}{dt}\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\,dz\,dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{\frac{du}{dt}+\frac{du}{dy}\frac{dy}{dt}\right\}dz\,dx$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\left\{\frac{dz}{dt}+\frac{dz}{dy}\frac{dy}{dt}\right\}dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\frac{dx}{dt}\,dz. \qquad \text{(C)}$$

In equation (A), Art. 377, first change $u$ into $\int_{y_0}^{y_1} u\,dy$, and reduce the first member by equation (1), Art. 375, with $x$ put for $t$, and $y$ for $x$; then change $u$ into $\int_{z_0}^{z_1} u$, reducing the first member by equation (2), Art. 376, with $x$ put for $t$, and $z$ for $x$; and lastly, change $u$ into $ut$, performing the differentiation in the first member. Then transposing, we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} u\frac{dt}{dx}\,dy\,dx = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} ut\,dy - \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} ut\frac{dy}{dx}\,dx$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{\left(\frac{du}{dx}+\frac{du}{dz}\frac{dz}{dx}\right)t + u\frac{dt}{dz}\frac{dz}{dx}\right\}dy\,dx. \qquad \text{(D)}$$

Next, in equation (A), Art. 377, put $y$ for $x$, thus:

$$\int_{y_0}^{y_1} \frac{du}{dy}\, dy = \Big/_{y_0}^{y_1} u.$$

In this equation first change $u$ into $\Big/_{z_0}^{z_1} u$, reducing the first member by equation (2), Art. 376, with $y$ put for $t$, and $z$ for $x$; then change $u$ into $ut$, perform the indicated differentiations in the first member, and finally prefix an integral sign followed by $dx$ to both members, as we evidently have a right to do. Then transposing, we have

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} u\frac{dt}{dy}\,dy\,dx = \int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\Big/_{z_0}^{z_1} ut\,dx$$

$$-\int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1}\left\{\left(\frac{du}{dy}+\frac{du}{dz}\frac{dz}{dy}\right)t + u\frac{dt}{dz}\frac{dz}{dy}\right\}dy\,dx. \qquad \text{(E)}$$

In equation (A), Art. 377, change $u$ into $\Big/_{y_0}^{y_1} u$, and reduce the first member by equation (2), Art, 376, with $x$ put for $t$, and $y$ for $x$. Then change $u$ into $\int_{z_0}^{z_1} u\,dz$, and reduce the first member by equation (1), Art. 375. Lastly, changing $u$ into $ut$, and transposing, we obtain

$$\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1} u\frac{dt}{dx}\,dz\,dx =$$

$$\Big/_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1} ut\,dz - \int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\Big/_{z_0}^{z_1} ut\left\{\frac{dz}{dx}+\frac{dz}{dy}\frac{dy}{dx}\right\}dx$$

$$-\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{\left(\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}\right)t + u\frac{dt}{dy}\frac{dy}{dx}\right\}dz\,dx. \qquad \text{(F)}$$

**455.** Suppose now that we had, as we shall presently have, occasion to consider such expressions as

$$U=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} V dy\, dx, \qquad U=\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1} V dz\, dx,$$

or

$$U=\Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} V dz\, dy,$$

which we may call mixed expressions, $V$ denoting any function of the independent variables $x$, $y$ and $z$, and of any quantities dependent upon them. Then it is evident that equation (9) must still hold for $\delta U$; and therefore, by putting $V$ for $u$ in formulæ (A), (B) and (C), we obtain at once the equations

$$\delta U=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1}\left\{\delta V+\frac{dV}{dz}\delta z\right\}dy\, dx$$

$$+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\Big/_{z_0}^{z_1} V\delta y\, dx+\Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1} V\,\delta x\, dy, \tag{G}$$

$$\delta U=\Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{\delta V+\frac{dV}{dx}\delta x\right\}dz\, dy$$

$$+\Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1}\left\{V\delta z+V\frac{dz}{dx}\delta x\right\}dy$$

$$+\Big/_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{V\delta y+V\frac{dy}{dx}\delta x\right\}dz, \tag{H}$$

$$\delta U=\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{\delta V+\frac{dV}{dy}\delta y\right\}dz\, dx$$

$$+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{V\delta z+V\frac{dz}{dy}\delta y\right\}dx+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}V\,\delta x\,dz, \quad \text{(I)}$$

where $\delta x$, $\delta y$ and $\delta z$ are synonymous with $Dx$, $Dy$ and $Dz$, and $\delta V$ is obtained upon the supposition that $x, y$ and $z$ do not vary, while $\frac{dV}{dx}$, $\frac{dV}{dy}$ and $\frac{dV}{dz}$ are the partial differential coefficients of $V$ taken only under the supposition that $x, y$ and $z$ enter $V$ explicitly.

Moreover, if in any of the formulæ ($A$), ($B$), ($C$), ($D$), etc., we wish to substitute in the first member but one limit of the variable, we need merely substitute in the second member the same single for the double limits of that particular variable, none of the other substitutions being in any way affected.

## Problem LXVII.

**456.** *It is required to determine the form which must be assumed by a surface of given area in order that it may enclose a maximum volume.*

This is only the most general statement of the problem, particular cases of which were discussed in Prob. LXIV. and also in Prob. XVI. Denote the volume by $v$. Then, although we might express $v$ by a double integral, we shall, for greater generality, write

$$v=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}dz\,dy\,dx. \quad (1)$$

Now the surface $S$ which bounds this volume, and which is to retain an invariable area, must be supposed, as usual in triple integrals, to consist of the six limiting faces $C_0$, $C_1$, $B_0$, $B_1$, $A_0$ and $A_1$, $S$ being their sum. Moreover, these several faces are expressed by the equations

$$\left.\begin{aligned}
C_0 &= \int_{x_0}^{x_1}\int_{y_0}^{y_1} \Big/^{z_0} \sqrt{1+p^2+q^2}\,dy\,dx, \\
C_1 &= \int_{x_0}^{x_1}\int_{y_0}^{y_1} \Big/^{z_1} \sqrt{1+p^2+q^2}\,dy\,dx, \\
B_0 &= \int_{x_0}^{x_1} \Big/^{y_0} \int_{z_0}^{z_1} \sqrt{1+y'^2}\,dz\,dx, \\
B_1 &= \int_{x_0}^{x_1} \Big/^{y_1} \int_{z_0}^{z_1} \sqrt{1+y'^2}\,dz\,dx, \\
A_0 &= \Big/^{x_0} \int_{y_0}^{y_1}\int_{z_0}^{z_1} dz\,dy, \\
A_1 &= \Big/^{x_1} \int_{y_0}^{y_1}\int_{z_0}^{z_1} dz\,dy,
\end{aligned}\right\} \quad (2)$$

where $y' = \frac{dy}{dx}$, $p = \frac{dz}{dx}$ and $q = \frac{dz}{dy}$. But assuming $-a$ as the constant multiplier, we are, by the method of Euler, to maximize absolutely the expression

$$U = v - a(A_0 + A_1 + B_0 + B_1 + C_0 + C_1) = v - aS. \quad (3)$$

Here the limits of the integrals are subject to no explicit restrictions, and it might therefore appear as though no maximum could be possible; but problems of relative maxima and minima do not always require any additional explicit restrictions upon the limits, the implicit restriction that the variations should be so taken as to render one or more of the integrals constant being sufficient for a definite solution. In the present case it will be found that the fact that $S$ is to maintain always the same value constitutes an implicit restriction, which is sufficient.

**457.** From formulæ (H), (I) and (G), by putting for $V$ 1 in the first, $\sqrt{1+y'^2}$ in the second, and $\sqrt{1+p^2+q^2}$ in the third, and substituting but one limit, say the inferior, in the first member, we obtain

$$\left.\begin{aligned}
\delta A_0 &= \int^{x_0}\int_{y_0}^{y_0}\int_{z_0}^{z_1}\{\delta z + p\delta x\}dy + \int^{x_0}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\{\delta y + y'\delta x\}dz,\\
\delta B_0 &= \int_{x_0}^{x_1}\int^{y_0}\int_{z_0}^{z_1}\frac{y'}{\sqrt{1+y'^2}}\,\delta y'\,dz\,dx\\
&\quad + \int_{x_0}^{x_1}\int^{y_0}\int_{z_0}^{z_1}\left\{\sqrt{1+y'^2}\,\delta z + \sqrt{1+y'^2}\,q\,\delta y\right\}dx\\
&\quad + \int_{x_0}^{x_1}\int^{y_0}\int_{z_0}^{z_1}\sqrt{1+y'^2}\,\delta x\,dz,\\
\delta C_0 &= \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int^{z_0}\left\{\frac{p}{\sqrt{1+p^2+q^2}}\,\delta p\right.\\
&\qquad\qquad\qquad\left. + \frac{q}{\sqrt{1+p^2+q^2}}\,\delta q\right\}dy\,dx\\
&\quad + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int^{z_0}\sqrt{1+p^2+q^2}\,\delta y\,dx\\
&\quad + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int^{z_0}\sqrt{1+p^2+q^2}\,\delta x\,dy,
\end{aligned}\right\}\quad(4)$$

and similar equations with the single suffixes changed from 0 to 1 will evidently hold for $\delta A_1$, $\delta B_1$ and $\delta C_1$.

We now transform by equations (D), (E) and (F) the terms containing $\delta p$, $\delta q$ and $\delta y'$, which may be done by substituting in these equations, respectively,

$$u = \frac{p}{\sqrt{1+p^2+q^2}},\qquad \frac{dt}{dx} = \frac{d\delta z}{dx} = \delta p,$$

$$u = \frac{q}{\sqrt{1+p^2+q^2}},\qquad \frac{dt}{dy} = \frac{d\delta z}{dy} = \delta q,$$

$$u = \frac{y'}{\sqrt{1+y'^2}},\qquad \frac{dt}{dx} = \frac{d\delta y}{dx} = \delta y',$$

observing that

$$\frac{dt}{dz} = \frac{d\delta z}{dz} = 0 \qquad\text{and}\qquad \frac{dt}{dy} = \frac{d\delta y}{dy} = 0,$$

and also that

$$\frac{du}{dx} + \frac{du}{dz} p = \frac{du}{dx}, \qquad \frac{du}{dy} + \frac{du}{dz} q = \frac{du}{dy},$$

and

$$\frac{du}{dx} + \frac{du}{dy} y' = \frac{du}{dx},$$

where the differentials in the second members are merely total. Having effected these transformations, and taken also the variations of $v$ in (1), we shall have

$$\left.\begin{aligned}
\delta v &= \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\delta z\,dy\,dx + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\delta y\,dz\,dx \\
&\quad + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\delta x\,dz\,dy, \\
\delta A_0 &= \int^{x_0}\int_{y_0}^{y_1}\int_{z_0}^{z_1}(\delta z + p\delta x)dy + \int^{x_0}\int_{y_0}^{y_1}\int_{z_0}^{z_1}(\delta y + y'\delta x)dz, \\
\delta B_0 &= -\int_{x_0}^{x_1}\int^{y_0}\int_{z_0}^{z_1}\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\,\delta y\,dz\,dx \\
&\quad + \int_{x_0}^{x_1}\int^{y_0}\int_{z_0}^{z_1}\left\{\sqrt{1+y'^2}\,\delta z + \frac{q-py'}{\sqrt{1+y'^2}}\,\delta y\right\}dx \\
&\quad + \int_{x_0}^{x_1}\int^{y_0}\int_{z_0}^{z_1}\left\{\sqrt{1+y'^2}\,\delta x + \frac{y'}{\sqrt{1+y'^2}}\,\delta y\right\}dz, \\
\delta C_0 &= -\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int^{z_0}\left\{\frac{d}{dx}\frac{p}{\sqrt{1+p^2+q^2}}\right. \\
&\qquad\qquad \left. + \frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2}}\right\}\delta z\,dy\,dx \\
&\quad + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int^{z_0}\left\{\frac{q-py'}{\sqrt{1+p^2+q^2}}\,\delta z + \sqrt{1+p^2+q^2}\,\delta y\right\}dx \\
&\quad + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int^{z_0}\left\{\frac{p}{\sqrt{1+p^2+q^2}}\,\delta z + \sqrt{1+p^2+q^2}\,\delta x\right\}dy,
\end{aligned}\right\} \quad (5)$$

and it is evident that $\delta A_1$, $\delta B_1$ and $\delta C_1$ will be expressed by precisely the same equations with every single suffix 0 changed to 1.

**458.** But, as appears from (3), we must, to obtain $\delta U$, multiply the last three, or rather the last six, of equations (5) by $-a$, and add the result to $\delta v$. If, then, in this equation we resolve all the signs of substitution, and then bring together the terms which contain like variations, and are affected by like substitutions (which M. Sarrus does), $\delta U$ will consist of thirty distinct terms, six holding throughout the six limiting faces, and the remaining twenty-four referring to what might be termed the twenty-four edges of these six faces, each actual edge of the body being regarded as belonging to either of two faces; and these thirty terms are independent.

But following the device of Moigno and Lindelöf, we may write $\delta U$ in the following condensed manner:

$$\delta U = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\Big/_{z_0}^{z_1}\left\{1 \pm \left(\frac{d}{dx}\frac{ap}{\sqrt{1+p^2+q^2}}\right.\right.$$

$$\left.\left.+\frac{d}{dy}\frac{aq}{\sqrt{1+p^2+q^2}}\right)\right\}\delta z\,dy\,dx$$

$$+\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{1 \pm \frac{d}{dx}\frac{ay'}{\sqrt{1+y'^2}}\right\}\delta y\,dz\,dx$$

$$+\Big/_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\delta x\,dz\,dy$$

$$\mp a\int_{x_0}^{x_1}\Big/_{y_0}^{y_1}\Big/_{z_0}^{z_1}\left\{\left(\sqrt{1+y'^2} \pm \frac{q-py'}{\sqrt{1+p^2+q^2}}\right)\delta z\right.$$

$$\left.+\left(\frac{q-py'}{\sqrt{1+y'^2}} \pm \sqrt{1+p^2+q^2}\right)\delta y\right\}dx$$

$$\mp a\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{\left(1 \pm \frac{p}{\sqrt{1+p^2+q^2}}\right)\delta z\right.$$

$$\left.+\left(p \pm \sqrt{1+p^2+q^2}\right)\delta x\right\}dy$$

$$\mp a\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}\left\{\left(1 \pm \frac{y'}{\sqrt{1+y'^2}}\right)\delta y\right.$$

$$\left.+\left(y' \pm \sqrt{1+y'^2}\right)\delta x\right\}dz = 0, \qquad (6)$$

where the signs of substitution denote throughout $\delta U$ the same series of operations as before; the sign $\pm$ in the first three terms denoting that at the first substitution the upper, and at the second the lower is to be taken; the same sign in the last three terms signifying that the upper is to be taken when the quantities substituted have the same suffix, and the lower when they have not, while these results must still, as the sign $-$ or $+$ indicates, be multiplied by $-1$ or $+1$, according as the quantity above the left-hand sign of substitution has the suffix 1 or 0. But the reader who may prefer can easily write out the thirty terms from (5), and verify directly the last and the following assertions.

**459.** Now equating to zero the coefficient of $\delta z$ in the first term of (6), we have

$$\frac{d}{dx}\frac{p}{\sqrt{1+p^2+q^2}}+\frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2}} = \mp\frac{1}{a}.$$

Or denoting by $r$ and $r'$ the principal radii of curvature, the last equation is equivalent to

$$\frac{1}{r}+\frac{1}{r'}=\frac{1}{a} \quad \text{and} \quad \frac{1}{r}+\frac{1}{r'}=-\frac{1}{a}; \qquad (7)$$

the first holding throughout the face $C_1$, and the second throughout the face $C_0$. Then the two equations (7) show that $C_0$ and $C_1$ have their mean curvatures constant, but turn their convexities in opposite directions.

Equating to zero the coefficients of $\delta z$ in the fourth term, we have

$$\frac{q - py'}{\sqrt{(1 + p^2 + q^2)(1 + y'^2)}} = \mp 1, \tag{8}$$

which involves the four equations relative to the intersection of the $C$'s and $B$'s, the negative sign holding for the edges $C_0 B_0$ and $C_1 B_1$, and the positive sign for the edges $C_1 B_0$ and $C_0 B_1$. But the first member of (8) equals the cosine of the angle made with each other by the two surfaces along their common intersection; and since this cosine is unity, we infer that the $B$'s and $C$'s are always tangent or accord along their common edges.

We observe, also, that equation (8) will cause the coefficient of $\delta y$ in the same fourth term to vanish without giving any additional equations. For since the $B$'s and $C$'s are tangent, $p$ and $q$ will have the same meaning in both along their intersection: thus ten equations have been considered.

Equating to zero the coefficient of $\delta z$ in the fifth term of (6), we have, for the intersections of the $A$'s and $C$'s,

$$\frac{p}{\sqrt{1 + p^2 + q^2}} = \mp 1; \tag{9}$$

$-1$ when the suffixes are alike, and $+1$ when they are unlike. Equation (9) denotes that the faces $C_0$ and $C_1$ must, along their intersections with the planes $A_0$ and $A_1$, be normal to the axis of $x$; that is, they must be tangent to or accord with these planes.

Then we observe, as before, that (9) causes the coefficient of $\delta x$ in this same fifth term to vanish without giving rise to

any additional equations. Thus, then, eight terms more have been caused to disappear.

**460.** Now having equated to zero the second term in (6), which is relative to the cylinders $B_0$ and $B_1$, and remembering that $\delta y$ must remain constant along any particular generatrix, but is independent for each, we shall obtain

$$\int_{z_0}^{z_1}\left\{1 \pm \frac{d}{dx}\frac{ay'}{\sqrt{1+y'^2}}\right\}dz = 0, \qquad (10)$$

which, with the positive sign, holds for any generatrix of $B_1$, and with the negative sign for the corresponding generatrix of $B_0$. But as the integration in (10) is to be effected regarding $x$, $y$ and $y'$ as constant, we have

$$\left\{1 \pm \frac{d}{dx}\frac{ay}{\sqrt{1+y'^2}}\right\}(z_1 - z_0) = 0. \qquad (11)$$

Equating the first factor to zero, we would obtain a cylinder of radius $a$, the limits of $z$ being wholly undetermined. But neglecting for the present this supposition, we must have $z_0 = z_1$; that is, $B_0$ and $B_1$ vanish or reduce to mere edges.

The condition $z_0 = z_1$ will cause also the first member in the last term of (6) to vanish without giving any new equations; so that thus six more terms in all disappear.

Equating to zero the third term in (6), and remembering that $\delta x_0$ and $\delta x_1$ are two independent constants, we have

$$\int_{y_0}^{y_1}\int_{z_0}^{z_1} dz\,dy = 0, \qquad (12)$$

an equation which involves two, as it holds for either of the faces $A_0$ or $A_1$, and shows at once that these two plane faces must also disappear.

Then (12) will cause the last member in the last term of (6), which is relative to the four intersections of the $A$'s and

$B$'s, to vanish without giving any new equations. Thus all the terms in (6) have been caused to vanish severally.

**461.** If we admit into the solution the cylinder with radius $a$, or for $A_0$ and $A_1$ any edge perpendicular to the axis of $x$ (which is probably admissible), we cannot say that all the conditions of the question could be satisfied. But if we assume $B_0$ and $B_1$ to become mere edges, and $A_0$ and $A_1$ to become points only, the volume in question must be entirely enclosed by the curved faces $C_0$ and $C_1$.

Moreover, from what has been shown it will appear that these two faces must be respectively perpendicular to the plane of $xy$ along their common intersection, and they must therefore meet in such a manner as to coalesce and to form one and the same surface, which will be given by the equation derived from (7),

$$\left(\frac{1}{r}+\frac{1}{r'}\right)^2=\frac{1}{a^2}. \tag{13}$$

Now the sphere of radius $2a$ will evidently satisfy all these conditions. But in order to exclude all other hypotheses, it would still be necessary to show that the sphere is the only admissible solution obtainable by equating to zero the terms of the first order in $\delta U$. But the proof of this fact has never yet been obtained by analyses; and even if it could be, it would still be necessary to show that the sphere would cause the terms of the second order to become always positive, or else those of some other even order to become so, the preceding having reduced to zero; and this would present a new and probably an insurmountable difficulty. Moreover, as we take the entire sphere, we shall be obliged to deal with some quantities which will become infinite; which fact might of itself throw some doubt upon our investigations. But although the complete discussion of this problem appears to be beyond the power of the present methods of analysis, we are assured from other considerations that the sphere is its true and only solution.

# CHAPTER IV.

## APPLICATION OF THE CALCULUS OF VARIATIONS TO DETERMINING THE CONDITIONS WHICH WILL RENDER A FUNCTION INTEGRABLE ONE OR MORE TIMES.

### SECTION I.

*CASE IN WHICH THERE IS BUT ONE INDEPENDENT VARIABLE.*

### Problem LXVIII.

**462.** *Suppose we seek by the calculus of variations to maximize or minimize the expression*

$$U = \int_{x_0}^{x_1} \left\{ \frac{y'}{y} - \frac{xy'^2}{y^2} + \frac{xy''}{y} \right\} dx = \int_{x_0}^{x_1} V dx. \qquad (1)$$

Then returning to our former notation, we shall have

$$N = V_y = -\frac{y'}{y^2} + \frac{2xy'^2}{y^3} - \frac{xy''}{y^2},$$

$$P = V_{y'} = \frac{1}{y} - \frac{2xy'}{y^2}, \qquad Q = \frac{x}{y},$$

$$\frac{dP}{dx} = P' = -\frac{3y'}{y^2} - \frac{2xy''}{y^2} + \frac{4xy'^2}{y^3},$$

$$\frac{d^2Q}{dx^2} = Q'' = -\frac{2y'}{y^2} - \frac{xy''}{y^2} + \frac{2xy'^2}{y^3};$$

so that the equation

$$M = N - P' + Q'' = 0$$

will reduce to $0 = 0$; that is, $M$ will vanish of itself, or identically; so that we obtain no equation from which we can derive a general solution, and have left merely the terms at the limits, which may be written

$$\Big/_{x_0}^{x_1} L = \Big/_{x_0}^{x_1} \left\{ (P - Q')\,\delta y + Q\delta y' \right\}$$

$$= \Big/_{x_0}^{x_1} \left\{ -\frac{xy'}{y^2}\,\delta y + \frac{x}{y}\,\delta y' \right\} = 0. \tag{2}$$

**463.** Now in seeking to explain this anomaly, we observe that $V$ may be written

$$V = \frac{y(xy'' + y') - xy'y'}{y^2}.$$

Whence we see that

$$\int V dx = \frac{xy'}{y} + c, \qquad \text{and} \qquad U = \Big/_{x_0}^{x_1} \frac{xy'}{y}. \tag{3}$$

Thus it appears that $U$ can in this case be freed from the sign of integration, and that the discussion of the conditions which will render it a maximum or a minimum does not, strictly speaking, belong to the calculus of variations; and we can readily show that whenever $U$ is integrable, $M$ must vanish identically. For assume the equation $U = \int_{x_0}^{x_1} V dx$, where $V$ is any function of $x, y, y', \ldots y^{(n)}$, but is of such a form that $Vdx$ shall be immediately integrable; that is, integrable independently of any relations which may hold between $x$ and $y$. Then we know that by definite integration $U$ may be written

$$U = \Big/_{x_0}^{x_1} f(x, y, y', \ldots y^{(n-1)}), \tag{4}$$

which shows that $U$ depends solely upon the limiting values of $x, y, y'$, etc., the relations between $x$ and $y$ being altogether in our power. Now if in $U$, before integration, we vary $y, y'$, etc., but suppose these quantities at the limits to remain fixed, $U$ will undergo no change; that is, $\delta U$ will vanish; and because $\delta y$, $\delta y'$, etc., are zero at the limits, $\delta U$ can, as usual, be reduced to the form

$$\delta U = \int_{x_0}^{x_1} M\,\delta y\,dx = 0, \tag{5}$$

to satisfy which, since $\delta y$ is in our power, $M$ must vanish. But unless $M$ vanish identically, we shall, by equating it to zero, have an equation which, if it be integrable, will determine $y$ as a function of $x$, or, if not integrable, will establish an implicit or differential relation between them, both of which are contrary to the conditions of the question. If, therefore, $Vdx$ be integrable immediately—that is, without assuming any particular relation, either explicit or implicit, between $x$ and $y$—$M$ must vanish identically.

**464.** Conversely, if, $U$ and $V$ having the same meaning as before, we find $M$ to vanish identically, we may conclude that $Vdx$ is immediately integrable. For we see that $\delta U$ will in this case consist of the terms at the limits only, as in equation (2), so that we infer that $U$ must depend solely upon the values which $x, y, y'$, etc., may have at the limits; and hence that $U$ must in reality be a function of these quantities only, which, so long as $y$ is wholly in our power, could not be true unless $Vdx$ were immediately integrable.

This mode of reasoning would seem to be sufficiently conclusive; nevertheless it is not so regarded by Prof. Todhunter, and the reader will find in his Integral Calculus, Art. 382, an attempt at a more rigorous demonstration.

## Problem LXIX.

**465.** *$V$ having the same form as before, it is required to determine the conditions which will render $V$ immediately integrable any number of times, $m$.*

First assume $m$ to be 2, and we have

$$\int \left\{ \int V dx \right\} dx = x \int V dx - \int x V dx; \qquad (1)$$

and hence, to insure that $V$ shall be twice immediately integrable, we must have both $V$ and $Vx$ immediately integrable; and conversely, if these quantities be immediately integrable once, $V$ will be immediately twice.

Now the first condition will give

$$N - P' + Q'' - \text{etc.} = M = 0, \qquad (2)$$

which must be true identically; while putting $v$ for $Vx$, the second condition will give, in like manner,

$$v_y - v_{y'}{}' + v_{y''}{}'' - \text{etc.} = 0, \qquad (3)$$

which must also be true identically. But (3) may be replaced by another equation, thus:

$$v_y = x V_y, \qquad v_{y'} = x V_{y'}, \qquad v_{y''} = x V_{y''}, \text{ etc.,}$$

$$v_{y'}{}' = x V_{y'}{}' + V_{y'}, \qquad v_{y''}{}'' = x V_{y''}{}'' + 2 V_{y''}{}',$$

$$v_{y'''}{}''' = x V_{y'''}{}''' + 3 V_{y'''}{}'', \qquad \text{etc.}$$

Substituting these values in (3), and omitting those terms which are known to be zero by (2), we shall obtain

$$P - 2Q' + 3R'' - \text{etc.} = 0, \qquad (4)$$

which must be true identically.

Hence that $V$ may be immediately integrable twice, equations (2) and (4) must be identically true.

Now, in the more general case in which $m$ is any number less than $n$, it is generally shown in works on the integral calculus that, if we denote by $U$ the result of the integration of $V$ $m$ times, we may exhibit $U$ thus:

$$U = \frac{1}{m-1}\left\{x^{m-1}\int V dx - (m-1)x^{m-2}\int x\, V dx\right.$$

$$\left.+\frac{(m-1)(m-2)}{1\cdot 2}x^{m-3}\int x^2 V dx - \text{etc.} \pm \int x^{m-1} V dx\right\}. \quad (5)$$

Whence it appears that to render $V$ integrable $m$ times it is necessary and sufficient that the quantities $V$, $Vx$, $Vx^2$, .... $Vx^{m-1}$ shall be severally integrable; and the equations arising from these conditions can be determined precisely as before. Thus if $m$ be 3, we shall find, in addition to equations (2) and (4), the identical equation

$$Q - \frac{3\cdot 2}{1\cdot 2}R' + \frac{4\cdot 3}{1\cdot 2}S'' - \text{etc.} = 0. \quad (6)$$

## Problem LXX.

**466.** *It is required to determine the conditions which will render $V dx$ immediately integrable, $V$ being any function of $x$ and the dependent variables $y$ and $z$, together with their differential coefficients with respect to $x$; that is, $y'$, $y''$, $z'$, $z''$, etc.*

Putting, as before, $U$ for the integral, and transforming $\delta U$, we shall obtain, as in Art. 303, a result which may be written

$$\delta U = L_1 - L_0 + \int_{x_0}^{x_1} M\, \delta y\, dx + \int_{x_0}^{x_1} N\, \delta z\, dx, \quad (1)$$

where

$$M = V_y - V_{y'}' + V_{y''}'' - \text{etc.}, \qquad N = V_z - V_{z'}' + V_{z''}'' - \text{etc.} \quad (2)$$

Now, as before, we may suppose the limiting values of $y$, $y'$, $z$, $z'$, etc., to be fixed, so that $L_1$ and $L_0$ will vanish.

Moreover, $\delta y$ and $\delta z$ are entirely independent, so that $M$ and $N$ must severally vanish if $U$ is to depend solely upon the limiting values of $x, y, y', z, z'$, etc. But either or both the equations $M = 0$ and $N = 0$, unless they be identically true, would enable us to establish some explicit or implicit relation between $x, y$ and $z$, whereas we require that $Vdx$ shall be integrable irrespectively of any such relation, other than that $y$ and $z$ are to be regarded as functions of $y$ and $x$.

If $V$ were integrable $m$ times, it is easy to see that, as in Prob. LXIX., we must have $V$, $Vx$, $Vx^2$, etc., immediately integrable, since equation (5) of that problem requires merely that $V$ shall be a function of $x$, and it might, therefore, contain any number of dependent variables, $y$, $z$, $u$, and their differential coefficients with respect to $x$. Hence we should evidently obtain with such equations as (2), (4) and (6) similar equations in $z$.

Moreover, it will appear that for any other dependent variable $u$ which $V$ may contain, we shall require in addition a similar set of equations in $u$.

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## SECTION II.

### *CASE IN WHICH THERE ARE TWO INDEPENDENT VARIABLES.*

### Problem LXXI.

**467.** *It is required to determine the conditions which will render $\int\int Vdy\,dx$ reducible to a single integral, where $V$ is any function of $x$, $y$, $z$, $p$ and $q$, $x$ and $y$ being two independent variables, and $p$ and $q$ partial differentials of $z$ with respect to these variables.*

Denoting the definite integral by $U$, we know that after transformation $\delta U$ may be written

$$\delta U = L + \int_{x_0}^{x_1}\int_{y_0}^{y_1} M\,\delta z\,dy\,dx, \tag{1}$$

where $L$, although consisting of simple integrals, involves only terms which relate to the limits of the integration; and by supposing $z$ to be unvaried along the lines $x = x_0$ and $x = x_1$, we can make $L$ consist only of quantities which are functions of $x$, and the variations of such quantities.

Now we know that if we regard all the quantities at the limits as fixed, $L$ will vanish, so that if $U$ can be reduced to a single integral depending upon these quantities only, $M$ must vanish; and if this reduction is to be possible without determining $z$ as some function, explicit or implicit, of $x$ and $y$, $M$ must vanish identically, otherwise the equation $M = 0$ will establish some such relation.

On the other hand, if $M$ vanish identically, $\delta U$ will reduce to $L$, and we infer that, as it depends solely upon quantities at the limits, $U$ is immediately reducible to a single integral.

**468.** Now we can determine what form $V$ must have to render this reduction possible. For

$$M = V_z - V_p' - V_{q,}; \tag{2}$$

and if $M$ is to vanish identically, it is evident, in the first place, that all terms containing $p$, $q$, $r$, $s$ and $t$ must vanish. But we have seen (Art. 428) that when, and only when, $V$ is of the form $f_1 + f_2 p + f_3 q$, the equation $M = 0$ will fail to rise above an order $n - 2$; that is, $2 - 2$.

Such, then, must be the form of $V$; but that $M$ may entirely vanish it will be necessary, in addition to this, that $f_1$, $f_2$ and $f_3$, which are all functions of $x$, $y$ and $z$ only, shall be subject to a certain additional relation. For we have

$$V_z = \frac{df_1}{dz}, \qquad V_p = f_2, \qquad V_q = f_3,$$

$$V_p' = f_2' = \frac{df_2}{dx} + \frac{df_2}{dz}p, \qquad V_{q,} = f_{3,} = \frac{df_3}{dy} + \frac{df_3}{dz}q;$$

so that the equation $M = 0$, after rejecting terms containing differential coefficients of $z$, gives

$$\frac{df_1}{dz} - \frac{df_2}{dx} - \frac{df_3}{dy} = 0,$$

and the $f$'s must be so related as to render this equation also identically true.

**469.** Although not very rigorously demonstrated, the foregoing are all the leading theorems relative to this subject, and it would be unprofitable to pursue it further. For while the calculus of variations gives us the means of determining whether or not $V$ is immediately integrable, it does not of itself indicate the method of effecting the integration; and this method is what we wish chiefly to know.

The theorems given in the preceding problems relative to this subject, which is often called the *theory of integrability*, or the *conditions of integrability*, can be established without the aid of the calculus of variations, but less easily.

The reader who may wish to pursue this subject further is referred to the treatise on the calculus of variations by Prof. Jellett, Chap. X., and also to Todhunter's History of the Calc. of Var., Chap. XVII.

# CHAPTER V.

## HISTORICAL SKETCH OF THE RISE AND PROGRESS OF THE CALCULUS OF VARIATIONS.

**470.** Questions of maxima and minima were among the first to occupy the attention of mathematicians after the invention of the differential or fluctionary calculus, which, according to Woodhouse, occurred about the year 1684, or three years prior to the publication of the *Principia*. The ordinary calculus was not, however, given to the world at once in a single treatise, but was developed gradually in essays, in communications to learned societies and journals, and in letters between men of science.

The first question considered, of that particular species of maxima and minima which forms the chief subject of the calculus of variations, appears to have been that of the *solid of minimum resistance;* and this was first proposed by Newton in the *Principia*. But although Newton was the first to consider a question belonging to the calculus of variations, no importance seems to have been attached to this problem either by himself or his contemporaries, and it did not become at that time the subject of discussion.

The true beginning of our science dates from the month of June, 1696, when John Bernoulli, Professor of Mathematics at Groningen, proposed in the *Acta Eruditorum* (or Doings of the Learned), a work then published at Leipsic, and at that time the chief medium of communication between men of science and letters, the following problem:

"PROBLEMA NOVUM.*

"Ad cujus solutionem mathematici invitantur.

"Datis in plano verticali duobus punctis $A$ et $B$, assignare mobili $M$ viam $AMB$, per quam gravitate sua descendens, et moveri incipiens a puncto $A$, brevissimo tempore perveniat ad alterum punctum $B$."

This problem engaged at once the attention of Leibnitz, James Bernoulli, brother of John, and Professor of Mathematics at Basle, and the Marquis de l'Hôpital, the first two of whom appear to have solved the problem within the allotted time, which was six months. Leibnitz at once forwarded his solution to the proposer, asking that it might not be immediately published, in order that other mathematicians might be encouraged to attempt the problem; and he subsequently, as no solution appeared, requested that the period might be extended, a request with which John Bernoulli complied, and accordingly reannounced the problem in a programme dated at Groningen, January, 1697. Upon learning of this extension, James Bernoulli retained his solution, being desirous, as he stated, of investigating and adding to the problem certain others of a similar character; which he did, as we shall subsequently see.

In the *Acta* for the following May were published the solutions of the two Bernoullis, together with one by De l'Hôpital, the last being without demonstration. James is in advance of his brother; but as his solution is given by Woodhouse, it will here suffice to say that both brothers assume the principle that whatever maximum or minimum property is possessed by the whole of any required curve must be possessed also by every portion of the curve; and that therefore, if

* "A New Problem, to the solution of which mathematicians are invited.

"Given two points $A$ and $B$ in a vertical plane, to find for the movable (particle) $M$, the path $AMB$, descending along which by its own gravity, and beginning to be urged from the point $A$, it may in the shortest time reach the other point $B$."

we consider the required curve as a polygon of an infinite number of sides, it will be sufficient to consider two consecutive sides or elements. But this principle, while it enabled them to obtain in this case a correct result, is not universally true.

**471.** At the close of his paper James Bernoulli proposed two additional problems: first, to determine the curve of quickest descent from a given fixed point to a given vertical line; and second, among all curves having a given length and a given base, to find the curve such that a second curve, each of whose ordinates is some function of the corresponding ordinate or arc of the first, may contain a maximum or minimum area. But although in the first of these problems we have a particular case of the question subsequently considered by Lagrange as to what conditions must hold at the limits in maximizing or minimizing a definite integral, little appears to have been effected in this direction prior to the researches of that mathematician; so that we shall follow the second problem only.

The second case of this problem led to an acrimonious discussion between the Bernoullis which was little creditable to John. For still adhering to the principle mentioned at the close of the last article, which in this case fails, owing to the isoperimetrical property that the required curve must have a given length, he continually obtained erroneous results; nor would he frankly acknowledge his error after it had been indicated by his brother. No solution of this problem, however, appeared until James Bernoulli, in May, 1701, published his in the *Acta;* although a solution without demonstration had appeared in the *Acta* for the preceding June. In this demonstration three consecutive elements of the required curve are taken instead of two, and a rude mode of imposing the isoperimetrical condition is shown. A solution much more finished, but evidently borrowed from that of his

brother, was subsequently published by John Bernoulli in the *Memoirs of the Academy of Science* for 1718; and this solution may be found in Woodhouse's Isoperimetrical Problems. But no further advance, worthy of notice in a sketch like the present, appears to have been made until the advent of Euler.

**472.** The first contribution of this mathematician to our science was a memoir published in the sixth volume of the Ancient Commentaries of Petersburg, 1733. In this memoir Euler, taking up the subject where it had been left by the Bernoullis, divided his problems into classes: the first including those of absolute maxima and minima, the second those problems of relative maxima and minima in which but one restriction is imposed upon the variations, as in the problem of the brachistochrone when the path is to have a given length; while the third included those relative problems in which two restrictions are imposed, as when the brachistochrone path is to have a given length and also to enclose, with the aid of its extreme ordinates and the axis of $x$, a given area. The erroneous principle that the maximum or minimum property of the whole curve belongs to each portion also was virtually adopted, two consecutive elements only being considered in the problems of the first class, three in those of the second, and four in those of the third. Nevertheless, as he proceeded, he established and tabulated formulæ—twenty-four in all—for the various cases which might arise; and by this means was, at the close of this memoir, much in advance of the Bernoullis.

About the year 1740 or 1741 Euler summed up his researches in a second memoir published in the eighth volume of the Commentaries of Petersburg, the date of the volume being 1736. But this date proves nothing, as the same volume contains observations made in 1740. Euler had now discovered that when $V$ is a function of $x, y, y', \ldots . y^{(n)}$, his previous formulæ might be expressed by one more general formula,

which is still in use, and which we have denoted by the equation $M = 0$.

Also, the principle that the maximum or minimum property of the whole curve will belong equally to every portion was examined, and shown, in some cases at least, to be untrue; and lastly, some advance seems to have been made in the treatment of problems of relative maxima and minima. By this memoir the calculus of variation was greatly improved, and it contained, in fact, nearly all that its author ever discovered relative to this subject, although in a very confused and ill-arranged form.

In 1744 Euler published a tract entitled *Methodus inveniendi Lineas curvas Proprietate maximi minimive gaudentes.** This work, which was the most famous of its author relative to this subject, displays an amount of mathematical skill almost unrivalled. Problems were here, as at present, divided into two great classes, *absolute* and *relative*, and the treatment of the second was for the first time reduced to a perfect science by the discovery of the artifice still employed, and termed the Method of Euler. This work is also generally clear and systematic, containing an abundance of examples, including, with many others, most of those given by us in our first chapter; and at its close Euler had carried our science so far beyond the point which had been reached by the Bernoullis that he may, almost equally with Lagrange, be regarded as the author of the calculus of variations.†

Euler subsequently published in the tenth volume of the New Commentaries of Petersburg, 1766, two memoirs; in the first of which, entitled *Elementa Calculi Variationum* (or

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* "Method of finding curved lines enjoying the property of maximum or minimum."

† The preceding account has been taken almost entirely from Woodhouse's Isoperimetrical Problems; but for what follows we are indebted chiefly to Todhunter's History of the Progress of the Calculus of Variations during the Nineteenth Century.

Elements of the Calculus of Variations), he first gives our science its present name; while in the second he enunciates the theorem of Prob. LXVIII.: this being apparently the first investigation ever made relative to the conditions of integrability. Subsequently in his Integral Calculus, 1770, he extended the theorem to two dependent variables, as in Prob. LXX.; while Lexel, in 1771, established the principle of Prob. LXIX.

**473.** Prior to this period Lagrange, who is commonly regarded as the author of the calculus of variations, had commenced his labors. But as we have not space to consider his writings in detail, we shall merely indicate the particulars in which he improved our science.

First. Much ambiguity and awkwardness had previously arisen from the want of a good method of distinguishing between ordinary differentials and those differentials or increments which we now call variations. This difficulty Lagrange overcame by the invention of the symbol $\delta$, which, like $d$, could denote either an increment or an operation, and proved of the highest importance.

Secondly. The formula $M = 0$, and others, had been derived by Euler from geometrical conceptions by breaking the integral, or the required curve, into parts, and operating laboriously upon two, three, or four consecutive elements. But Lagrange, by deriving these formulæ by the methods now in use, shortened the processes of obtaining them, and placed our science upon its true analytical basis.

Thirdly. The formulæ of Euler determined merely the *nature* of the required curve, its extremities being supposed to be fixed. But Lagrange, in what he termed the definite equations, first gave the form and the interpretation of all those formulæ which are still employed when the extremities of the required curve are not fixed, and which we have called the equations or terms at the limits.

Fourthly. Lagrange invented that general method which is still employed, and known as the Method of Lagrange, and which enables us by the use of one or more indeterminate multipliers to discuss those cases in which the variables are connected by an implicit relation merely; that is, by a differential equation which is not integrable.

Lastly. Lagrange first attempted to extend the calculus of variations to the case of double integrals. This he did by discussing Prob. LVII., obtaining our equation (10), Art. 366, without considering the terms at the limits.

**474.** In the year 1810 appeared an English work, entitled "A Treatise on Isoperimetrical Problems and the Calculus of Variations," by Robert Woodhouse, A.M., F.R.S., Fellow of Caius College, Cambridge. The first five chapters of this work, which is a small octavo of 154 pages, with 9 pages of preface, are devoted to a careful history of the subject to the time of Lagrange, and are all that are now of any interest, the remaining three containing little history of importance.

The subject having been next, but not very successfully, treated by Lacroix in his *Traité du Calcul Différentiel et du Calcul Intégral*, second volume, second edition, 1814, there appeared two German works. The former, by E. H. Dirksen, which is a small quarto of 243 pages, with 8 pages of preface, was published at Berlin in 1823, and is entitled "Analytical Exhibition of the Calculus of Variations, with the Application of it to the Determination of Maxima and Minima." The latter is entitled "The Theory of Maxima and Minima," by Dr. Martin Ohm, Berlin, 1825, and is an octavo of 330, with a preface of 18 pages.

None of these works, however, extended the calculus of variations, and we now resume the history of its progress.

**475.** The first discussion of the discrimination of maxima and minima appears to have been undertaken by Legendre,

who, about the year 1787, elaborated the method already mentioned in Art. 187, and which was published the following year in the History of the Royal Academy of Science. This method was subsequently adopted by Lagrange, although he indicated the defect noticed in the above article. This method is explained in Todhunter's History of the Calculus of Variations.

Legendre seems also at the same time to have given the first instance of a discontinuous solution by showing in the discussion of Prob. XV. that it might be necessary for the required curve to be in part rectilinear.

In the *Memorie dell' Istituto Nazionale Italiano*, Vol. II. Part II., Bologna, 1810, Brunacci extended the method of discrimination to the case of a double integral; and although his method is open to the same objection as that of Legendre for single integrals, he succeeded in establishing all the conditions relative to $V_{pp}$, $V_{pq}$ and $V_{qq}$ mentioned in Art. 431; and their discovery appears to be due to him.

**476.** The variation of a double integral when the limits are also variable, the exhibition of the terms at the limits so as to determine the conditions which must there hold, and the variation of a multiple integral in general, were subjects which had not yet been investigated, and they next engaged the attention of mathematicians.

Three memoirs were published bearing more or less directly upon these subjects: the first by C. F. Gauss in 1829, the second by Poisson in 1831, and the third by Ostrogradsky in 1834. But while these writers effected much, they did not succeed in determining in a general manner the number and form of the equations which must subsist at the limits in the case of a double or triple integral.

**477.** In the seventeenth volume of Crelle's *Mathematical Journal*, 1837, appeared a memoir, entitled "On the Theory of the Calculus of Variations and of Differential Equations," by

C. G. Jacobi. This memoir, which purports to be an extract from a letter to Professor Enke, is devoted partly to the calculus of variations and partly to dynamics. In the first part Jacobi elaborated, but without demonstration, the theorem which bears his name; that is, he assumed the truth of our lemmas, not even giving the forms of the functions $A$, $A_1$, $A_2$, $B_1$, etc., although he determined the form of $u$, and merely touched upon the connection between $u$ and $v$. See Art. 174. This brevity rendered the theorem the subject of numerous commentaries, as we shall presently see.

Jacobi also touched upon the mode of transforming the terms of the first order in the variation of a double integral, but effected nothing of importance.

**478.** In 1841 were published three memoirs relative to Jacobi's theorem: the first two by V. A. Lebesgue and C. Delaunay, in the sixth volume of Liouville's *Journal of Mathematics*, and the last by Bertrand in the *Journal de l'Ecole Polytechnique*. The proof given by Delaunay is that which we have followed in our notes to Lemmas I. and II., and he has been generally followed by subsequent writers.

**479.** As, notwithstanding the labors of Gauss, Poisson, Ostrogradsky and Jacobi, no general method of treating the terms at the limits in the case of multiple integrals had yet been discovered, the Academy of Science, Paris, 1842, proposed for its mathematical prize the following subject: To find the limiting equations which must be combined with the indefinite equations in order to determine completely the maxima and minima of multiple integrals; the formulæ to be applied to triple integrals.

Of the four memoirs presented, that by Sarrus was adjudged worthy of the prize, while that by Delaunay received honorable mention; the examiners being Liouville, Sturm, Poinsot, Duhamel and Cauchy.

The memoir of Sarrus is entitled *Recherches sur le Calcul*

*des Variations*, and may be found in the tenth volume of the *Savants Étrangers*, 1846, and occupies 127 quarto pages. By means of his new symbol of substitution, Sarrus may be said to have solved the problem proposed by the Academy, and his memoir is one of the most important contributions of the century. But this sign of substitution as invented by Sarrus, besides having an inconvenient form, signified merely the substitution of a particular value of a variable for its general value, and his method therefore lacked brevity.

The treatment by Delaunay is much less general, assuming that in the case of double integrals the limiting cylinder or surface is to be continuous and closed. His memoir was published in the 29th cahier of the *Journal de l'École Polytechnique*, dated 1843, and seems to have been followed by all the writers on the calculus of variations subsequent to Moigno and Lindelöf.

**480.** The next advance was made by Cauchy in a memoir on the calculus of variations published in the third volume of his *Exercices d'analyse et de Physique Mathématique*, 1844, extending from page 50 to page 130. This memoir is little else than a reproduction of the investigations of Sarrus, but in it Cauchy effected much of the needed condensation by giving to the sign of substitution, like that of integration in a definite integral, the power of denoting substraction also, while its form was changed to that which we have adopted. For further particulars regarding this part of our subject the reader who does not wish to examine the original memoirs may consult the chapters on Sarrus and Cauchy in Todhunter's History of the Calculus of Variations.

**481.** We must next notice some systematic treatises which now appeared.

As a successor of Woodhouse there appeared "A Treatise on the Calculus of Variations," by Richard Abbatt, London, 1837. This is an octavo of 207 pages, with 11 pages of pre-

face, but is of no great importance at the present day, and could hardly be regarded as a complete treatise.

In the year 1850 appeared a work entitled "An Elementary Treatise on the Calculus of Variations," by the Rev. John Hewitt Jellett, A.M., Fellow of Trinity College and Professor of Natural Philosophy in the University of Dublin. This work, which is an octavo of 377 pages, with an introduction and preface of 20 pages, is one of the most important which have appeared in any language, and is not elementary as its title would imply. But Prof. Jellett had not, as he himself tells us, been able to peruse the memoir of M. Sarrus, while that of Cauchy is not mentioned by him at all. Hence his discussion of multiple integrals, in which he follows that memoir of Delaunay which received honorable mention by the French Academy, is defective, and cannot be recommended to the student.

Ohm's treatise was succeeded by a voluminous work by Dr. G. W. Strauch, entitled *Theorie und Anwendung des sogenannten Variationscalcul's*, Zurich, 1849. This treatise consists of two closely printed large octavo volumes, the first containing 499 pages, with 32 pages of preface, and the second 788 pages; and is chiefly valuable for its great number of carefully solved examples, and historical notes, although, as might be expected, much of the matter has little or no connection with the calculus of variations. Strauch does not exhibit the theorem of Jacobi, although he generally examines the terms of the second order, employing the method of Legendre and Lagrange without even noticing its defect. He is also like Jellett deficient in the treatment of multiple integrals, not following the method of Sarrus and Cauchy. Strauch subsequently, in 1856, presented to the Academy of Sciences in Vienna a memoir entitled *Anwendung des sogenannten Variationscalcul's auf zweifache und dreifache Integrale*, and published in the 16th volume of the *Denkschriften* of the Academy, 1859, where it occupies 156 large quarto pages; and in

this memoir he even declares that Sarrus and Cauchy did not solve the problem proposed by the French Academy. His own memoir is, however, of no importance.

In a few years appeared another German work by Dr. Stegmann, entitled *Lehrbuch der Variationsrechnung und ihrer Anwendung bei Untersuchungen über das Maximum und Minimum*, Kassel, 1854. This is an octavo of 417 pages, with 16 pages of preface, but is not so rich in examples as is the treatise by Strauch, while it possesses the two defects mentioned in connection with that treatise.

Prof. Bruun published in the Russian language "A Manual of the Calculus of Variations," Odessa, 1848, which is, according to Prof. Todhunter, an octavo of 195 pages.

We may mention, finally, that Prof. Price in the second volume of his Treatise on Infinitesimal Calculus, Oxford, 1854, devoted more than 100 pages to our science, explaining the theorem of Jacobi, and touching upon the subject of double integrals.

**482.** After the publication of the three memoirs mentioned in Art. 478, the subject of the discrimination of maxima and minima was not considered for about ten years, after which it was resumed earnestly by mathematicians in papers, some of which we will next mention.

In the third volume of Tortolini's *Annali di Scienze Mathematiche e Fisiche*, 1852, appeared an article of more than 40 pages by Prof. G. Mainardi, claiming, but without good reason, to exhibit a new method of discriminating maxima and minima. But he also extended Jacobi's theorem to double integrals, and his method has been followed by us in treating this subject.

In the same volume appeared a short article on the same subject by Prof. F. Brioschi. Mainardi had indicated the value of the theory of determinants in connection with the exhibition of the terms of the second order, and Brioschi

employed it freely, this being apparently the first attempt to apply determinants to this subject.

There next appeared a quarto pamphlet of 20 pages regarding Jacobi's theorem, entitled *Untersuchungen über Variations-rechnung. Inaugural-Dissertation von Dr. Friedrich Eisenlohr*, Manheim, 1853.

The subject was next considered in a work entitled "On the Criteria for Maxima and Minima in Problems of the Calculus of Variations," which was presented by Spitzer to the Academy of Sciences at Vienna in 1854. This work consists of two memoirs occupying together more than 135 pages, the first being published in the 12th and the second in the 14th volume of the *Sitzungsberichte* of the Academy, and to these memoirs we are indebted for the exceptions which we have noticed in connection with Jacobi's theorem. But Mainardi and Spitzer did not confine themselves to the development of Jacobi's theorem, but sought rather to establish new methods of their own, both of which are, according to Prof. Todhunter, "Legendre's method improved by additions borrowed from Jacobi."

In the 54th volume of Crelle's *Mathematical Journal*, 1857, appeared a memoir by Otto Hesse, entitled "On the Criteria for the Maxima and Minima of Single Integrals," extending over pages 227–273. Hesse confines himself exclusively to the application of Jacobi's theorem to single integrals involving only one dependent variable, and his memoir is the most elaborate which has yet appeared regarding this subject. See Arts. 184, 186.

In the 55th volume of Crelle's *Mathematical Journal*, 1858, appeared a memoir by A. Clebsch, entitled "On the Reduction of the Second Variation to its Simplest Form," and extending over pages 254–273. The object of Clebsch was to generalize the theorem of Jacobi, and to supply investigations like those of Hesse for the case in which the single integral contains several dependent variables with or without connecting

equations, and also for multiple integrals. The former point had not, so far as the author knows, been hitherto discussed, but the latter had been considered by Mainardi. The subject of multiple integrals is resumed by him in a third memoir, entitled "On the Second Variation of Multiple Integrals," and published in the 56th volume of Crelle's *Mathematical Journal*, 1859, where it extends over pages 122–148. His second memoir is "On those Problems in the Calculus of Variations which involve only one Independent Variable," and is in the same volume which contains his first memoir.

**483.** We now come to a most valuable work, entitled "A History of the Progress of the Calculus of Variations during the Nineteenth Century," by I. Todhunter, M.A., Fellow and Principal Mathematical Lecturer of St. John's College, Cambridge. Macmillan & Co., London, 1861. This volume is a large octavo of 530 pages, with 10 pages of preface, and, taken together with the first five chapters of Woodhouse, furnishes a complete history of our subject. But in addition to the mathematics necessary to the historical sketches, much of which has been superseded by better methods, Prof. Todhunter has frequently introduced these better methods, and has given such other investigations of his own that his work contains nearly all the matter necessary to form a modern treatise, although, from the nature of the case, it is so arranged as to be of little service to the reader who is not already tolerably familiar with the calculus of variations. We append the subjects of the seventeen chapters: Chap. I., Lagrange, Lacroix; II., Dirksen, Ohm; III., Gauss; IV., Poisson; V., Ostrogradsky; VI., Delaunay; VII., Sarrus; VIII., Cauchy; IX., Legendre, Brunacci, Jacobi; X., Commentators on Jacobi; XI., On Jacobi's Memoir; XII., Miscellaneous Memoirs; XIII., Systematic Treatises; XIV., Minor Treatises; XV., XVI., Miscellaneous Articles; XVII., Conditions of Integrability. The last chapter is a complete history of the

subject from the earliest times as it had not been mentioned by Woodhouse, nor had its history been given by any previous writer.

**484.** A few months subsequently, but during the same year, 1861, appeared the last systematic treatise, the *Calcul des Variations*, by Moigno and Lindelöf. But the title-page of this work, which is a small octavo of 352 pages, with 20 additional pages of preface, introduction, etc., presents it in the beginning as merely the fourth volume of the *Leçons de Calcul Différentiel et de Calcul Intégral*, by M. l'Abbé Moigno, the distinctive title following subsequently. According to Moigno, the chief credit of this work, which is the only complete treatise in the French language, belongs to his colleague, M. Lindelöf, then a young professor from the university of Helsingfors in Finland, who had made the calculus of variations a specialty, and who gave Moigno freely the benefit of his knowledge.

This treatise was the first to present a satisfactory account of the conditions which must hold at the limits when we wish to maximize or minimize the double or triple integral. But although the methods followed are substantially those of Sarrus and Cauchy, the authors have, in many cases, greatly simplified the formulæ of their masters; and to this portion of the *Calcul des Variations* the present author is almost entirely indebted for the discussions which have been presented in Chapter III.; although the view of variations adopted by Moigno and Lindelöf is that followed by Sarrus, and which has been explained in Section VI. Chap. III.

**485.** It had long been known that a discontinuous solution might become necessary in certain problems. But although particular cases had been discussed by Legendre and others, nothing resembling a general theory of such solutions had yet been propounded.

In the *Philosophical Magazine* for June, 1866, Prof. I. Tod-

hunter first announced the principle that variations might be of restricted sign, thus rendering it unnecessary for the equation $M = 0$ to hold throughout $U$; and this may be regarded as the fundamental principle of the theory in question. This discovery appears to have been due mainly to the difficulties presented by the consideration of Prob. XVI.

In 1869 this subject was proposed at Cambridge for the Adams Essay, and elicited from Prof. Todhunter in 1871 the prize essay, which, with slight alteration, was published in the same year by Macmillan & Co., under the title "Researches in the Calculus of Variations."

This work, which is an octavo of 278 pages, and 8 pages of introduction, is certainly the most important original contribution which our science has received since the appearance of the essay of Sarrus, inasmuch as, in it, the author, while discussing incidentally many other points of interest, did for the theory of discontinuous solutions, what Sarrus did for that of multiple integrals. The case of single integrals only is discussed, and these are, with a few exceptions, supposed to involve but one dependent variable. The theory is, however, abundantly illustrated by examples; and we cannot too strongly recommend the work to our readers, since, from it, we have derived most of what we have presented in Section IX, Chap. I.

# NOTES.

## NOTE TO LEMMA I.

To establish this theorem, which belongs entirely to the differential calculus, we shall employ the symbolic language, or, as it is sometimes called, the calculus of operations. (See De Morgan's Diff. and Integ. Calc., page 751; also Boole's Diff. Eqs., Chap. XVI.)

Let $d$ denote differentiations with respect to $k$ only, and $D$ with respect to $\delta y$ only, both $k$ and $\delta y$ being regarded as functions of $x$, and the differentials with regard to $x$ being total. Then any order of total differential of any function of $k\,\delta y$ may be written $(d+D)^n$ of that function. Now putting $v$ for pair (4), we have

$$\begin{aligned} v &= [(d+D)^m D^l \pm (d+D)^l D^m]k\,\delta y \\ &= (d+D)^l D^l[(d+D)^{m-l} \pm D^{m-l}]k\,\delta y \\ &= X^l[(d+D)^{m-l} \pm D^{m-l}]k\,\delta y, \end{aligned} \tag{1}$$

where

$$X = (d+D)D = dD + D^2. \tag{2}$$

It must, however, be remembered that $X$ does not denote a quantity, but merely a mode of differentiation, and that seeming exponents as 2, $n$, etc., do not indicate powers, but the number of times that a certain mode of differentiation is performed.

Now from (2) we have

$$D^2 + dD + \frac{d^2}{4} = \frac{d^2}{4} + X = \frac{1}{4}(d^2 + 4X), \tag{3}$$

or

$$\left(D + \frac{d}{2}\right)^2 = \frac{1}{4}(d^2 + 4X); \tag{4}$$

the first member of (4) denoting differentiation twice according to a predetermined method, the second differential having been rendered perfect by the addition of another differential, just as the square is, in quadratics, by the addition of a square. Hence, solving as in quadratics, we obtain

$$D = \frac{1}{2}\left\{ - d \pm (d^2 + 4X)^{\frac{1}{2}} \right\} \tag{5}$$

and

$$d + D = \frac{1}{2}\left\{ d \pm (d^2 + 4X)^{\frac{1}{2}} \right\} = \frac{1}{2}(d \pm r), \tag{6}$$

$r$ denoting also a mode of differentiation only.

Now put $n$ for $m - l$. Then in (1), by the use of (5) and (6), we have

$$(d + D)^{m-l} \pm D^{m-l} = (d + D)^n \pm D^n = \frac{1}{2^n}\left\{ (d \pm r)^n \pm (- d \pm r)^n \right\}, \tag{7}$$

the positive or negative sign being used according as $n$ is even or odd.

If we first suppose $n$ to be even, and expand both binomials by the binomial theorem, and add the results, then, since each term which does not cancel becomes double, we shall, after multiplying $\frac{1}{2^n}$ by 2, have

$$\frac{1}{2^{n-1}}\left\{ d^n + \frac{n(n-1)}{2} d^{n-1} r^2 + \text{etc.} \right\}. \tag{8}$$

Let us next suppose $n$ to be odd. Then the development will assume the same form, because the sign connecting the two binomials will be negative, so that we shall have always

$$(d + D)^{m-l} \pm D^{m-l} = \frac{1}{2^{n-1}}\left\{ d^n + \frac{n(n-1)}{2} d^{n-2} r^2 + \text{etc.} \right\}. \tag{9}$$

But since $r^2 = d^2 + 4X$, we will suppose the values of $r^2$, $r^4$, etc., in (9) to have been found and arranged according to the ascending superscripts of $X$. Then there will occur in $r^2$, $r^4$, $r^6$, etc., one term involving $d$, but not $X$; and this term, when combined with its component outside of $r$, which will involve $d$, will always become $d^n$ multiplied by some function of $n$. Therefore the development may be written

$$(d + D)^{m-l} \pm D^{m-l} = ad^n + bd^{n-2}X + cd^{n-4}X^2 + \text{etc.}, \tag{10}$$

where $a$, $b$, $c$, etc., are functions of $n$ and numbers merely. Hence from (1) we have

$$v = (ad^n X^l + bd^{n-2} X^{l+1} + cd^{n-4} X^{l+2} + \text{etc.}) k \delta y$$

$$= (d + D)^l D^l d^n ak \delta y + (d+D)^{l+1} D^{l+1} d^{n-2} bk \delta y$$

$$+ (d+D)^{l+2} D^{l+2} d^{n-4} ck \delta y + \text{etc.} \qquad (11)$$

We have now only to abandon the symbols of separation by performing the operations which they indicate, thus:

$$D^l d^n ak \delta y = \frac{d^n}{dx^n} ak \frac{d^l \delta y}{dx^l} = c_l \frac{d^l \delta y}{dx^l} = c_l \delta y^{(l)}; \qquad (12)$$

and since $d + D$ denotes total differentiation, we have

$$(d + D)^l c_l \delta y^{(l)} = \frac{d^l}{dx^l} c_l \delta y^{(l)}.$$

Proceeding in like manner with the other terms in (11), we shall finally obtain

$$v = \frac{d^l}{dx^l} c_l \delta y^{(l)} + \frac{d^{l+1}}{dx^{l+1}} c_{l+1} \delta y^{(l+1)} + \text{etc.}, \qquad (13)$$

where

$$c_l = \frac{d^n}{dx^n} ak, \qquad c_{l+1} = \frac{d^{n-2}}{dx^{n-2}} bk, \qquad c_{l+2} = \frac{d^{n-4}}{dx^{n-4}} ck, \qquad (14)$$

and

$$\left.\begin{aligned} a &= \frac{1}{2^{n-1}} \left\{ 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}} + \text{etc.} \right\}, \\ b &= \frac{1}{2^{n-1}} \left\{ 4 \frac{n(n-1)}{2} + 8 \frac{n(n-1)(n-2)(n-3)}{\underline{|4}} + \text{etc.} \right\}, \\ c &= \frac{1}{2^{n-1}} \left\{ 16 \frac{n(n-1)(n-2)(n-3)}{\underline{|4}} + \text{etc.} \right\}. \end{aligned}\right\} \qquad (15)$$

Now the application of (13) is in reality simple. For we see that for any given value of $n$ its number of terms must always be the same as that of equation (9). Hence there will be but one term when $n$ is 1, two when $n$ is 2 or 3, three when $n$ is 4 or 5, etc. Moreover, it will be found from (15) that $a$ is always unity, and that $b = n$ when $n$ has any value from 2 to 5 inclusive, and that $c = 2$ when $n$ is 4, and is 5 when $n$ is 5.

## NOTE TO LEMMA II.

The integration may be effected as follows: Multiply equation (1) by $ut$, and subtract the product from (2). Then we have

$$U = u\frac{d}{dx}A_1(ut)' + u\frac{d^2}{dx^2}A_2(ut)'' + \text{etc.} - ut\frac{d}{dx}A_1u' - ut\frac{d^2}{dx^2}A_2u'' - \text{etc.}$$

$$= \Sigma\left\{u\frac{d^m}{dx^m}A_m(ut)^{(m)} - ut\frac{d^m}{dx^m}A_m u^{(m)}\right\}. \tag{1}$$

Now we know that if $P$ and $Q$ be any two quantities, we shall have

$$PQ^{(n)} = (PQ)^{(n)} - n(P'Q)^{(n-1)} + \frac{n(n-1)}{2}(P''Q)^{(n-2)} - \text{etc.} \tag{2}$$

For let $d$ denote differentiations with respect to $Q$, and $D$ with respect to $P$. Then we shall have

$$PQ^{(n)} = d^n PQ, \qquad (PQ)^{(n)} = (d+D)^n PQ, \qquad (P'Q)^{(n-1)} = (d+D)^{n-1}DPQ, \text{ etc.}$$

Now in the cases which we shall consider $n$ will be a not large positive integer, and it will therefore readily appear by trial that

$$d^n = (d+D)^n - n(d+D)^{n-1}D + \frac{n(n-1)}{2}(d+D)^{n-2}D^2 - \text{etc.} \pm D^n.$$

Hence if we select any term of (1), as

$$u\frac{d^m}{dx^m}A_m(ut)^{(m)} \quad \text{or} \quad u[A_m(ut)^{(m)}]^{(m)},$$

and put $u$ for $P$, and the other factor for $Q^{(n)}$, we obtain, by the use of (2),

$$u\left[A_m(ut)^{(m)}\right]^{(m)} = \left[uA_m(ut)^{(m)}\right]^{(m)} - m\left[u'A_m(ut)^{(m)}\right]^{(m-1)}$$

$$+\frac{m(m-1)}{2}\left[u''A_m(ut)^{(m)}\right]^{(m-2)} - \text{etc.} \tag{3}$$

But

$$(ut)^{(m)} = ut^{(m)} + mu't^{(m-1)} + \frac{m(m-1)}{2}u''t^{(m-2)} + \text{etc.} \tag{4}$$

Substituting this value in (3), each term may expand into a series, each series having the superscript of the term from which it was derived. Now consider any series, and let its superscript be $p$, so that it must be the $m-p+1$ in order, and every term must contain the factor $u^{(m-p)}A_m$. Now for an individual term, take that in this series whose order is $m-q+1$, or $q+1$ when we begin at the last. This term will be of the form

$$M(u^{(m-p)}A_m u^{(m-q)}t^{(q)})^{(p)} = M(kt^{(q)})^{(p)} = M\frac{d^p}{dx^p}\left(k\frac{d^q t}{dx^q}\right), \tag{5}$$

where $k = u^{(m-p)}A_m u^{(m-q)}$, and

$$M = \pm\frac{m(m-1)\ldots\ldots(p+1)}{1, 2, 3\ldots\ldots(m-p)}\cdot\frac{m(m-1)\ldots\ldots(q+1)}{1, 2, 3\ldots\ldots(m-q)}, \tag{6}$$

while $p$ and $q$ must be some positive integer, or zero, and $m$ some positive integer. Now if $p$ and $q$ be unequal, there will, supposing $p$ greater than $q$, certainly arise in the series whose superscript is $q$ a term of the form

$$M\frac{d^q}{dx^q}\left(k\frac{d^p t}{dx^p}\right),$$

the signs being like or unlike according as $p-q$ is even or odd. Hence, by the theorem of the preceding note, all the terms in $\Sigma u\frac{d^m}{dx^m}A_m(ut)^{(m)}$ in which $p$ and $q$ are unequal may be transformed, so that by adding those in which $p$ and $q$ are equal, which have already the required form, we may write

$$\Sigma u\frac{d^m}{dx^m}A_m(ut)^{(m)} = Bt + \frac{d}{dx}B_1t' + \frac{d^2}{dx^2}B_2t'' + \text{etc.} \tag{7}$$

But if the differentiation indicated in the first member of (7) were performed, it is evident that the terms which would contain $t$ undifferentiated would be

$$\Sigma u(A_m u^{(m)})^{(m)}t = Bt = \Sigma ut(A_m u^{(m)})^{(m)}. \tag{8}$$

Hence it appears from (1) that all the terms containing $t$ undifferentiated will disappear from $U$, and we shall, therefore, have

$$U = \frac{d}{dx}B_1t' + \frac{d^2}{dx^2}B_2t'' + \text{etc.} \tag{9}$$

Therefore

$$\int U\,dx = B_1t' + \frac{d}{dx}B_2t'' + \text{etc.} \tag{10}$$

## NOTE TO ART. 369.

Let $A$, $B$ and $C$ denote the angles made by the normal with $x$, $y$ and $z$ respectively, and let the Greek letter $\xi$ (xi or $x$) denote the angle made with the plane of $xz$ by the plane which contains the normal and is parallel to $z$, and $\eta$ (eta or $\acute{e}$) be $l \tan \frac{C}{2}$ or $l \tan c$, so that, $e$ being the Napierian base, we shall have

$$e^\eta = \tan \frac{C}{2} = \tan c.$$

Our object is now to change in (10) the independent variables from $x$ and $y$ to $\xi$ and $\eta$. We have

$$\sin C = \sin 2c = 2 \sin c \cos c = \frac{2 \sin c}{\cos c} \cos^2 c = \frac{2 \tan c}{1 + \tan^2 c} = \frac{2e^\eta}{1 + e^{2\eta}}, \tag{1}$$

$$\cos C = \cos 2c = \cos^2 c - \sin^2 c = \left(\frac{\cos^2 c}{\sin^2 c} - 1\right) \sin^2 c$$

$$= \frac{\dfrac{1}{\tan^2 c} - 1}{1 + \dfrac{1}{\tan^2 c}} = \frac{e^{-2\eta} - 1}{e^{-2\eta} + 1}. \tag{2}$$

These equations will enable us to express $\sin C$ and $\cos C$ in terms of $\cos \eta$ and $\tan \eta$; but the process will evidently involve the theory of hyperbolic sines, cosines, etc. We may recall from this theory the following formulæ, putting $i$ for $\sqrt{-1}$:

$$\sin u = \frac{1}{2i}(e^{iu} - e^{-iu}), \qquad \cos u = \frac{1}{2}(e^{iu} + e^{-iu}), \qquad i \tan u = \frac{e^{2iu} - 1}{e^{2iu} + 1}. \tag{3}$$

These formulæ occur in De Morgan's Diff. and Integ. Calc., pages 114 and 119, except that we have put $u$ for $\theta$ on the first page, and for $x$ on the second. Now if in the second and third of these equations we put $i\eta$ for $u$, we shall have

$$\cos i\eta = \frac{1}{2}(e^{-\eta} + e^\eta) = \frac{e^{2\eta} + 1}{2e^\eta}, \qquad i \tan i\eta = \frac{e^{-2\eta} - 1}{e^{-2\eta} + 1} \tag{4}$$

Hence from (1) and (2) we have

$$\sin C = \frac{1}{\cos i\eta}, \qquad \cos C = i \tan i\eta. \tag{5}$$

Now it is evident that

$$\cos A = \sin C \cos \xi = \frac{\cos \xi}{\cos i\eta}, \qquad \cos B = \sin C \sin \xi = \frac{\sin \xi}{\cos i\eta}. \tag{6}$$

Hence equation (10) Art. 366, now becomes

$$\left.\begin{aligned} &\frac{d}{dx}\frac{\cos \xi}{\cos i\eta} + \frac{d}{dy}\frac{\sin \xi}{\cos i\eta} = 0, \\ \text{or} \quad &-\sin \xi \frac{d\xi}{dx} + \cos \xi \frac{d\xi}{dy} + i \tan i\eta \left\{ \cos \xi \frac{d\eta}{dx} + \sin \xi \frac{d\eta}{dy} \right\} = 0, \end{aligned}\right\} \tag{7}$$

in which we must next determine the values of the partial differential coefficients $\frac{d\xi}{dx}$, $\frac{d\xi}{dy}$, $\frac{d\eta}{dx}$ and $\frac{d\eta}{dy}$.

Let $X$, $Y$ and $Z$ be the co-ordinates of any point of any tangent plane to the required surface, and $D$ the distance of this plane from the origin. Then we shall have

$$X \cos A + Y \cos B + Z \cos C = D;$$

and substituting in this the values of $\cos A$, $\cos B$ and $\cos C$, and multiplying by $\cos i\eta$, we have

$$X \cos \xi + Y \sin \xi + Z i \sin i\eta = D \cos i\eta = -\zeta, \tag{8}$$

the Greek letter zeta being used for convenience only. But since (8) represents the equation of any tangent plane, and every point of the required surface lies on one of these planes, the equation of this surface may be written

$$x \cos \xi + y \sin \xi + zi \sin i\eta = -\zeta, \tag{9}$$

where $\zeta$ is no longer constant, but must be such a function of $\xi$ and $\eta$ that the variable $D$ may always have the meaning just assigned. Moreover, we may regard every point of the required surface as lying at the intersection of three tangent planes drawn indefinitely near, so that in the second $\xi$ may become $\xi + d\xi$, $\eta$ remaining unchanged; and in the third $\eta$ may become $\eta + d\eta$, $\xi$ remaining unchanged; $\xi$ and $\eta$ themselves belonging to the first. Hence we have a right to differentiate (9) with regard to $\xi$ and $\eta$ separately, treating $x$, $y$ and $z$ as constants. Performing this operation, we have

$$x \sin \xi - y \cos \xi = \frac{d\zeta}{d\xi}, \qquad z \cos i\eta = \frac{d\zeta}{d\eta}. \tag{10}$$

Next assume, for brevity,

$$u=\zeta+i\tan i\eta\frac{d\zeta}{d\eta}+\frac{d^2\zeta}{d\xi^2},\qquad v=\frac{d^2\zeta}{d\xi d\eta},\qquad w=i\tan i\eta\frac{d\zeta}{d\eta}+\frac{d^2\zeta}{d\eta^2}. \tag{11}$$

We have also, from (9) and the last of equations (10),

$$-(x\cos\xi+y\sin\xi)=\zeta+zi\sin i\eta,\qquad zi\sin i\eta=i\tan i\eta\frac{d\zeta}{d\eta}. \tag{12}$$

We will now differentiate equations (10) and (11), reducing by means of the equations whose first members are the bracketed quantities, and supposing the last to have been obtained first, so as to employ it in reducing the first. Thus we shall have

$$\left.\begin{aligned}
dx\cos\xi+dy\sin\xi &= [x\sin\xi-y\cos\xi]\,d\xi-i\sin i\eta\,dz\\
&\quad+[z\cos i\eta]\,d\eta-\frac{d\zeta}{d\xi}d\xi-\frac{d\zeta}{d\eta}d\eta\\
&=-d[zi\sin i\eta]=-i\tan i\eta\,(v\,d\xi+w\,d\eta),\\
dx\sin\xi-dy\cos\xi &= -[x\cos\xi+y\sin\xi]\,d\xi+\frac{d^2\zeta}{d\xi^2}d\xi+\frac{d^2\zeta}{d\xi\,d\eta}d\eta\\
&=\zeta d\xi+[zi\sin i\eta]\,d\xi+\frac{d^2\zeta}{d\xi^2}d\xi+\frac{d^2\zeta}{d\xi\,d\eta}d\eta=u\,d\xi+v\,d\eta,\\
dz\cos i\eta &= [zi\sin i\eta]\,d\eta+\frac{d^2\zeta}{d\xi\,d\zeta}d\xi+\frac{d^2\zeta}{d\eta^2}d\eta=v\,d\xi+w\,d\eta.
\end{aligned}\right\} \tag{13}$$

Now if in the first two of these equations we first make $dy$ zero in each and divide by $dx$, and then $dx$ zero in each and divide by $dy$, we shall obtain four equations, the first two of which will each contain $\frac{d\xi}{dx}$ and $\frac{d\eta}{dx}$, and the last two of which will each contain $\frac{d\xi}{dy}$ and $\frac{d\eta}{dy}$; and these differentials will then become the partial differentials sought. Then finding the values of these differentials by common algebraic methods, we shall have

$$\frac{d\xi}{dx}=\frac{wi\tan i\eta\sin\xi-v\cos\xi}{i\tan i\eta\,(uw+v^2)}$$

$$\frac{d\eta}{dx}=\frac{vi\tan i\eta\sin\xi-u\cos\xi}{i\tan i\eta\,(uw+v^2)},$$

$$\frac{d\xi}{dy}=-\frac{wi\tan i\eta\cos\xi+v\sin\xi}{i\tan i\eta\,(uw+v^2)}$$

$$\frac{d\eta}{dy}=-\frac{vi\tan i\eta\cos\xi+u\sin\xi}{i\tan i\eta\,(uw+v^2)}.$$

If now we substitute these values in equation (7), observing, if we clear fractions, not to remove the imaginary quantity $i$ from the denominator, it will easily reduce to

$$u + w = 0. \tag{14}$$

This equation is not itself integrable, but we can easily obtain from it a more general expression, which can be integrated.

By making $d\xi$ and $d\eta$ alternately zero in the last of equations (13), we obtain

$$v = \frac{dz}{d\xi} \cos i\eta, \qquad w = \frac{dz}{d\eta} \cos i\eta, \tag{15}$$

where the differentials of $z$ have become partial, being taken with relation to $\xi$ and $\eta$ only, as separate independent variables. Now since (14) holds for every point of the required surface, its differential with relation to $\xi$ or $\eta$, or both, must be zero also. Let us therefore differentiate with respect to $\eta$ only. Then observing that

$$1 + \frac{d}{d\eta} i \tan i\eta = 1 + \frac{i^2}{\cos^2 i\eta} = 1 - \sec^2 i\eta = - \tan^2 i\eta = i \tan i\eta \cdot i \tan i\eta,$$

we have

$$\begin{aligned}
\frac{du}{d\eta} &= \frac{d\zeta}{d\eta}\left(1 + \frac{d}{d\eta} i \tan i\eta\right) + i \tan i\eta \frac{d^2\zeta}{d\eta^2} + \frac{d^3\zeta}{d\xi^2 d\eta} \\
&= i \tan i\eta \left(i \tan i\eta \frac{d\zeta}{d\eta} + \frac{d^2\zeta}{d\eta^2}\right) + \frac{d^3\zeta}{d\xi^2\, d\eta} = \frac{dv}{d\xi} + wi \tan i\eta \\
&= \frac{d^2z}{d\xi^2} \cos i\eta + \frac{dz}{d\eta} i \sin i\eta, \\
\frac{dw}{d\eta} &= \frac{d^2z}{d\eta^2} \cos i\eta - \frac{dz}{d\eta} i \sin i\eta.
\end{aligned}$$

Hence, by adding, we deduce from (14)

$$\frac{d^2z}{d\xi^2} + \frac{d^2z}{d\eta^2} = 0; \tag{16}$$

a partial differential of the second order, the complete integral of which is known to be

$$z = f(\xi + i\eta) + F(\xi - i\eta), \tag{17}$$

$f$ and $F$ denoting any functions whatever, real or imaginary. See De Morgan's Diff. and Integ. Calc., pp. 723, 719, putting $i$ for $a^2$, and $\eta$ for $t$. See also Boole's Diff. Eqs., Chap. XV.

It is evident that, having differentiated (14), the present integral is more general than the integral of that equation; but it includes the equation of the required surface, which, when the forms of $f$ and $F$ are assigned, must be deduced in the following manner. We have, from the last of equations (10),

$$\zeta = \int z \cos i\eta \, d\eta, \tag{18}$$

in which, by (17), $z$ will become a known function of $\xi$ and $\eta$. Then we shall obtain $\zeta$ by integrating with respect to $\eta$ only, observing to add to the result an arbitrary function of $\xi$, which function must be then so determined as to satisfy (14), otherwise the value of $\zeta$ will be too general. Now substituting this final value of $\zeta$ in equations (9) and (10), and then eliminating $\xi$ and $\eta$, we shall obtain the particular equation sought.

As an example, assume

$$f(\xi + i\eta) = \frac{1}{2}(a - ib)(\xi + i\eta),$$

$$F(\xi - i\eta) = \frac{1}{2}(a + ib)(\xi - i\eta).$$

Whence, by (17) and (18),

$$z = a\xi + b\eta,$$

$$\zeta = \int (a\xi + b\eta) \cos i\eta \, d\eta = -(a\xi + b\eta)\, i \sin i\eta - b \cos i\eta + X,$$

$X$ being any function of $\xi$; its first and second differential coefficients with respect to $\xi$ being $X'$ and $X''$. Now using this value of $\zeta$ in equations (11), we find easily

$$u = -b \cos i\eta + X'' + X, \qquad w = b \cos i\eta,$$

and

$$u + w = X'' + X = 0.$$

The integral of this equation is, as the reader can easily verify by differentiation, $X = c \cos \xi + c' \sin \xi$, $c$ and $c'$ being any arbitrary constants. If now we replace $X$ by this value, we shall have the true expression for $\zeta$ belonging to the particular surface sought. For greater simplicity, let us now suppose $c$ and $c'$ to become zero, so that $X$ will vanish also, and the value of $\zeta$ will become

$$\zeta = -(a\xi + b\eta)\, i \sin i\eta - b \cos i\eta = -zi \sin i\eta - b \cos i\eta. \tag{19}$$

Now resuming equation (9) and the first equation (10), and eliminating between them $x$ and $y$ alternately by multiplication, and substituting for $\zeta$ from (19), and for $\frac{d\zeta}{d\xi}$ its value $-ai\sin i\eta$, we shall obtain

$$\left.\begin{aligned} x &= b\cos i\eta\cos\xi - ai\sin i\eta\sin\xi, \\ y &= b\cos i\eta\sin\xi + ai\sin i\eta\cos\xi, \\ z &= a\xi + b\eta, \end{aligned}\right\} \qquad (20)$$

the last equation having been obtained before. Or if in the same equations we write

$$x = r\cos\omega, \qquad y = r\sin\omega,$$

observing that

$$x\cos\xi + y\sin\xi = r(\cos\omega\cos\xi + \sin\omega\sin\xi) = r\cos(\omega - \xi),$$

$$x\sin\xi - y\cos\xi = r(\cos\omega\sin\xi - \sin\omega\cos\xi) = r\sin(\omega - \xi),$$

we have in polar co-ordinates, $z$ remaining as before,

$$r\cos(\omega - \xi) = b\cos i\eta, \qquad r\sin(\omega - \xi) = ai\sin i\eta, \qquad z = a\xi + b\eta. \qquad (21)$$

If now, in equations (20) or (21), we can eliminate $\xi$ and $\eta$, we shall obtain a particular equation of the minimum surface. Suppose we make $a$ zero. Then (21) gives

$$\omega - \xi = 0 \quad \text{ and } \quad \eta = \frac{z}{b},$$

and by the first of equations (4) we shall have

$$r = b\cos\frac{iz}{b} = \frac{b}{2}\left(e^{\frac{z}{b}} + e^{-\frac{z}{b}}\right),$$

which is evidently the equation of the surface generated by the revolution of a catenary about the axis of $z$, that axis coinciding with the directrix; which is the same result as has been previously obtained. Making $b$ zero while $a$ is not, we have

$$\omega - \xi = \frac{\pi}{2}, \qquad \xi = \frac{z}{a}, \qquad \omega = \frac{\pi}{2} + \frac{z}{a},$$

the equation of a helicoid.

When $a$ and $b$ are any constants whatever, we can still eliminate $\xi$ and $\eta$ from (21). As the result merely is given by Moigno, we will here indicate the work without explanation. We have